ALAGAPPA UNIVERSITY

[Accredited with 'A+' Grade by NAAC (CGPA:3.64) in the Third Cycle and Graded as Category–I University by MHRD-UGC]

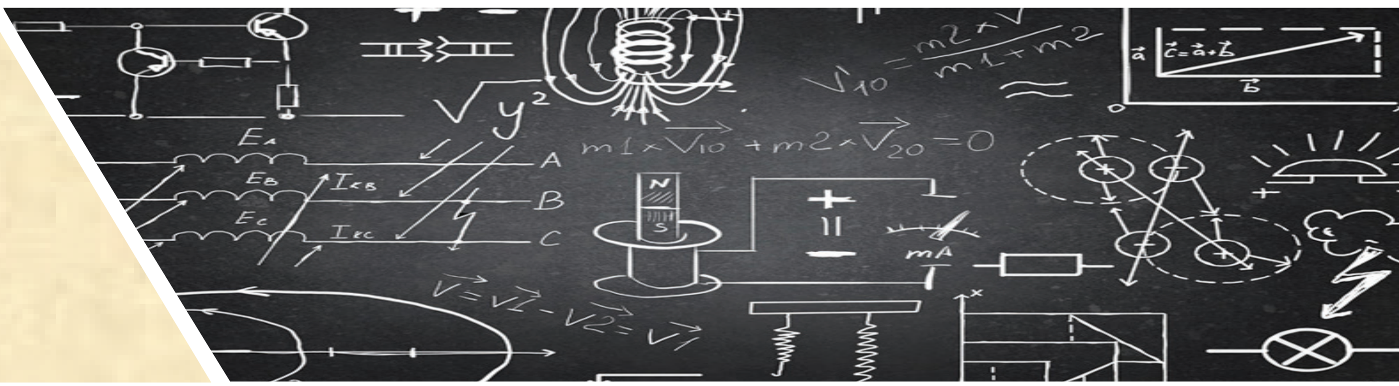
KARAIKUDI – 630 003

DIRECTORATE OF DISTANCE EDUCATION

Vallal Dr.RM.Alagappa Chettiar

M.Sc. [Physics]

345 11



CLASSICAL MECHANICS

I - Semester

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CLASSICAL MECHANICS

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Units (1-14)

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SYLLABI-BOOK MAPPING TABLE

Classical Mechanics

Syllabi	Mapping in Book
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BLOCK II: HAMILTONIAN METHODS **UNIT 4: Hamilton's Principle** Introduction – Hamilton's Principle – Hamilton's Principle for a Conservative System – Principle of Least Action. **UNIT 5: Hamilton-Jacobi Theory** Characteristic Function and Hamilton-Jacobi Equation. **UNIT 6: Canonical Transformations** Phase Space and Liouville's Theorem – Special Transformations – Lagrange Brackets – Poisson Bracket – Calculus of Variations.	**Unit 4:** Hamilton's Principle **(Pages 75-82);** **Unit 5:** Hamilton-Jacobi Theory **(Pages 83-106);** **Unit 6:** Canonical Transformations **(Pages 107-142)**
BLOCK III : KINEMATICS OF RIGID BODY MOTION AND SPECIAL THEORY OF REATIVITY **UNIT 7: Moment of Inertia** Moments and Products of Inertia – Moment of Inertia of a Body About Any Line – Through The Origin of Coordinate Frame – The Momental Ellipsoid – Rotation Coordinate Axes. **UNIT 8: Rigid Body Equations of Motion** Principal Axes and Principal Moments – Kinetic Energy of a Rigid Body Rotating About a Fixed Point – Angular Momentum of a Rigid Body – Eulerian Angles – The Compound Pendulum.	**Unit 7:** Moment of Inertia **(Pages 143-163);** **Unit 8:** Rigid Body Equations of Motion **(Pages 164-208);**
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CONTENTS

INTRODUCTION

NOTES

Classical mechanics describes the motion of macroscopic objects, from projectiles to parts of machinery, and astronomical objects, such as spacecraft, planets, stars and galaxies. If the present state of an object is known it is possible to predict by the laws of classical mechanics how it will move in the future (determinism) and how it has moved in the past (reversibility).The earliest development of classical mechanics is often referred to as Newtonian mechanics. It consists of the physical concepts employed by and the mathematical methods invented by Isaac Newton and Gottfried Wilhelm Leibniz and others in the 17th century to describe the motion of bodies under the influence of a system of forces. Later, more abstract methods were developed, leading to the reformulations of classical mechanics known as Lagrangian mechanics and Hamiltonian mechanics. These advances, made predominantly in the 18th and 19th centuries, extend substantially beyond Newton's work, particularly through their use of analytical mechanics. They are, with some modification, also used in all areas of modern physics.

Classical mechanics provides extremely accurate results when studying large objects that are not extremely massive and speeds not approaching the speed of light. When the objects being examined have about the size of an atom diameter, it becomes necessary to introduce the other major sub-field of mechanics: quantum mechanics. To describe velocities that are not small compared to the speed of light, special relativity is needed. In case that objects become extremely massive, general relativity becomes applicable. However, a number of modern sources do include relativistic mechanics into classical physics, which in their view represents classical mechanics in its most developed and accurate form.

This book, *Classical Mechanics*, is divided into four blocks which are further subdivided into fourteen units and principally deals with the mechanics of particles and rigid bodies. The first unit introduces the concept of Newton's laws of motion while Lagrange's equation have been discussed in the following unit. The third unit deals with the concept of Hamiltonian equation. Hamiltonian principle is focused on in the fourth unit while the fifth unit explains Hamilton-Jacobi theory. Canonical transformations have been discussed in the sixth unit while seventh unit deals with moment of inertia. Eighth units discusses the concept of rigid body equations of motion and the concept of special theory of relativity has been explained in the following unit. Lorentz transformation has been explained in the tenth unit while eleventh units focuses on one dimensional oscillators. Twelfth unit discusses normal modes while thirteenth unit introduces you to the general theory of small oscillators. The last units discusses linear triatomic molecule.

NOTES

The book follows the self-instructional mode wherein each unit begins with an 'Introduction' to the topic. The 'Objectives' are then outlined before going on to the presentation of the detailed content in a simple and structured format. 'Check Your Progress' questions are provided at regular intervals to test the student's understanding of the subject. A 'Summary', a list of 'Key Words' and a set of 'Self-Assessment Questions and Exercises' are provided at the end of each unit for effective recapitulation.

BLOCK-I
LAGRANGE AND HAMILTON EQUATIONS

NOTES

UNIT 1 NEWTON'S LAWS OF MOTION

Structure

1.0 INTRODUCTION

Newton's laws of motion are three physical laws that, together, laid the foundation for classical mechanics. They explain the relationship between a body and the forces acting upon it, and its motion in response to those forces. More precisely, the first law defines the force qualitatively, the second law offers a quantitative measure of the force, and the third asserts that a single isolated force does not exist. Kepler›s laws of planetary motion are three scientific laws describing the motion of planets around the Sun. Kepler's work improved the heliocentric theory of Nicolaus Copernicus, explaining how the planets' speeds varied, and using elliptical orbits rather than circular orbits with epicycles. In this unit you will describe Newton's laws of motion and laws of conservation. You will understand frames of reference, mass point and angular momentum. You will understand the Kepler's laws of planetary motion, and interpret stability of planetary orbits. Classification of dynamical system is also discussed in this unit.

1.1 OBJECTIVES

NOTES

After going through this unit, you will be able to:

- Explain Newton's laws of motion and laws of conservation
- Understand frames of reference, mass point and angular momentum
- Discuss Kepler's laws of planetary motion
- Interpret stability of planetary orbits
- Classify a dynamical system

1.2 NEWTON'S LAWS OF MOTION

Mechanics, which deals with the motion of objects around us, is one of the oldest studied branches of physics. It considers the conditions under which objects remain at rest and undergo motion when they are acted upon by internal and external forces without enquiring into the ultimate constitution of matter. Mechanics tells us how an object moves in a given situation and how the motion can be described.

Galileo, Huygens and Newton were the founders of mechanics which showed that objects moved according to certain rules. These rules were formulated in the form of the laws called the laws of motion. Newtonian mechanics is essentially the study of the consequences of the laws of motion formulated by Newton and published in the year 1687 in his *Philosophiae Naturalis Principia Mathematica* (the Principia).

The motion of bodies of macroscopic size can be described by considering them as assembly of particles, though rigorously speaking, it is not possible to identify such subdivision of matter into particles. A particle at any instant of time is described by three space coordinates together with a scalar constant called its mass. The particles in a body are held to move following Newton's laws which are stated as follows:

(i) A body continues to remain in a state of rest or moves with a uniform speed along a straight line unless subjected to an external force.

(*ii*) The rate of change of linear momentum of a body is proportional to the magnitude of the external force acting on it and takes place in the direction of that force.

(*iii*) The forces exerted by two bodies on each other are equal in magnitude and opposite in direction.

It is assumed that the terms mass, force, straight line, etc., have some intuitively understood meanings, and Newton's laws are logical statements of their inter-relations.

NOTES

Newtonian mechanics can be rightly termed as vectorial mechanics because it deals with quantities such as displacement, velocity, momentum, acceleration, and force, which are all vectors. This mechanics can be conveniently applied to relatively simple mechanical problems but at times it becomes difficult to apply the laws in complex problems particularly those involving constraints. To deal with such problems, alternative schemes have been developed which were initiated by Leibnitz. These are associated with the names of Euler, Lagrange, Hamilton, Poisson, Jacobi and others. These alternative formulations are referred to as analytical mechanics. Unlike Newtonian mechanics, the fundamental quantities involved in analytical mechanics are scalar rather than vector and the dynamical relations are obtained by a systematic process of differentiation. From the laws of mechanics postulated in their most general or analytical form emerge Newton's laws. We may emphasize that analytical mechanics (namely, the Lagrangian and the Hamiltonian formulations of mechanics) is not a new theory of mechanics but is alternative to Newtonian formulation. The difference in the two formulations lies in the process of formulating the equations of motion for a mechanical problem.

Newtonian mechanics together with the analytical mechanics has been found to describe correctly (in conformity with experiments) the motion of objects which are neither too big nor too small, and is referred to as classical mechanics. Though the range of validity of this mechanics has been extended enormously, it has been found to be completely inadequate for:

(*i*) The description of small scale phenomena of atomic and nuclear physics.

(*ii*) The description of motion of extremely big objects, namely the galaxies.

(*iii*) Phenomena involving moving objects with speeds approaching that of light.

In the case of small scale phenomena, quantum mechanics has superseded classical mechanics. Einstein's general theory of relativity (theory of gravitation) and special theory of relativity, respectively, describe the motion of big objects like galaxies and motion of objects moving with speeds comparable to that of light. Einstein's theories of relativity have found experimental confirmation.

It is important to note that both quantum mechanics and the theories of relativity may be considered as extensions of classical mechanics in the sense that they reproduce its results under appropriate limiting cases. Thus, these confirmed theories reinforce the correctness of classical mechanics within its vast range of validity and applicability. Indeed, classical mechanics is a remarkably successful theory which provides satisfactory account of the phenomena as diverse as the motion of a vehicle on the road, or the tides in ocean, or motion of artificial satellites and planets in orbits about the sun. Moreover, even outside this range, many of the results of classical mechanics still apply. In particular, the laws of conservation of linear momentum, angular momentum and energy which stem from the symmetry properties of space and time with respect to inertial frame of reference are, so far as we know today, of universal validity.

An important point to note about analytical mechanics, which is an integral part of classical mechanics, is that it is capable of generalization to situations such as classical field theory and quantum mechanics in which Newton's laws are inapplicable.

1.2.1 Fundamental ideas

(*i*) Frames of Reference

To describe the motion of a mechanical system (say, a body) it is necessary to specify its position in space as a function of time. It is meaningful to speak of the relative position of a body. The position of a body in space can be defined relative to the sun, the centre of the galaxy, etc.

Further, for describing the motion, besides stating the coordinates of the body relative to a suitable coordinate system it is necessary to specify the time instant at which the coordinates assume the given set of values. The concept of a single universal time is true only when the relative velocities of all bodies are small compared to the velocity of light. Newtonian mechanics is valid only under the above approximation.

If the coordinates stating the positions of moving bodies are fixed within the same system, a definite frame of reference is said to be defined.

In Newtonian mechanics, it is assumed that time instant in one frame of reference is the same in all other frames of reference. This is not a property of time, but of reference systems of bodies moving slowly relative to one another.

(*ii*) Mass Point or Mass

Formulation of the laws of motion uses an extremely convenient concept, namely that of a *mass point,* or *a particle.* A particle is defined as a body whose position in space can be exactly defined by three coordinates which may be cartesian, spherical polar, cylindrical, etc. This is an idealization and does not apply to any real body. However, it becomes reasonable when the motion of a body is sufficiently well defined by only the displacement of any of its points and is independent of its rotational motion and its deformations if any. This can be understood from the fact that while the motion of the earth round the sun is independent of its rotation about its axis, the trajectory of a bullet depends strongly on its rotational motion. Hence, that the earth can be approximated as a mass point more closely than a bullet. We find that dimensions of a body are immaterial if the body can be approximated as a mass point or a particle.

Considering the concept of a mass point, the law of motion, namely Newton's second law for a body, can be written as

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F} \quad (1)$$

In the above, $\vec{F}$ is the resultant of all the forces applied to the body and, $\frac{d^2\vec{r}}{dt^2}$ is the acceleration vector. The quantity m characterizes the mass point and is called its mass.

(*iii*) Force and Mass

Newton's second law expressed by Equation (1) is the physical definition of force. Underlying Equation (1) are a number of assumptions concerning the laws of motion. For example, the fact that Equation (1) involves a second derivative with respect to time requires for its solution the knowledge of two constants, namely the initial values of the coordinates and velocities, that is, the first derivative of coordinate which is sufficient to determine coordinates at all future instants of time. This fact can be deduced only from experimental data. Further, Equation (1) defines the mode of interaction between bodies, indicating that it is effected in the form of forces, imparting acceleration. This assertion, too, derives from a generalization of experimental data only.

Newtonian mechanics makes a limiting assumption about force because this mechanics, is valid only for velocities of bodies small in comparison with the velocity of light. The force depends upon the mutual configuration of the bodies at the instant to which the Equation (1) refers and is independent of their configurations at preceding instants.

Mass m characterizing a body, has dimensions or units of measurement very special one, unrelated to the units of displacement and time.

Experiments have established that mass is an additive quantity, meaning thereby that the quantity characterizing a body as a whole is equal to the sum of the these quantities for all its parts separately. The above principle of additivity of mass applies to bodies made up of different substances.

In Newtonian mechanics the mass of a given body is a constant quantity irrespective of the kind of motion it undergoes.

(*iv*) Laws of Conservation: Linear Momentum, Angular Momentum and Energy

The force acting on each particle of a system made up of N particles is the sum of two parts. One part is the internal force on the particle due to other particles in the system and the other part is the external force arising due to sources outside the system. We can hence write the total force acting on, say, the k^{th} particle as

NOTES

$$\vec{F_k} = \vec{F_k}^{(i)} + \vec{F_k}^{(e)} \tag{2}$$

The net force acting on the system as a whole is thus

$$\vec{F} = \sum_{k=1}^{N} \vec{F_k} = \sum_{k=1}^{N} \vec{F_k}^{(i)} + \sum_{k=1}^{N} \vec{F_k}^{(e)} \tag{3}$$

According to Newton's third law, internal forces are equal and opposite, i.e., force acting on the kth particle due to the jth particle $\vec{F_{kj}}$ is equal and opposite to that acting on the jth particle due to the kth particle $\vec{F_{jk}}\,(\vec{F_{kj}} = -\vec{F_{jk}})$. In view of this we must have

$$\sum_{k=1}^{N} \vec{F_k}^{(i)} = 0$$

and hence Equation (3) reduces to

$$\vec{F} = \sum_{k=1}^{N} \vec{F_k}^{(e)} \tag{4}$$

If $\vec{p_k}$ is the linear momentum of the k^{th} particle, according to Newton's second law,

we get

$$\vec{F_k}^{(e)} = \frac{d}{dt}(\vec{p_k}) \tag{5}$$

Combining Equation (4) and (5) gives

$$\vec{F} = \sum_{R=1}^{N} \vec{F_k}^{(e)} = \sum_{k} \frac{d}{dt}\left(\vec{p_k}\right) = \frac{d}{dt}\left[\sum \vec{p_k}\right]$$

or
$$\vec{F} = \frac{d\vec{P}}{dt} \tag{6}$$

In the above, $\vec{P} = \sum \vec{p_k}$ is the total linear momentum of the system.

If the sum of the external forces acting on the system is zero, i.e., if $\sum_{k=1}^{N} \vec{F_k}^{(e)} = 0$, then we obtain

$$\vec{F} = 0 \text{ and } \frac{d\vec{P}}{dt} = 0$$

or
$$\vec{P} = \text{a constant} \tag{7}$$

We thus find that if a system of particles is devoid of any net external force then the total linear momentum (magnitude as well as direction) of the system remains conserved throughout the motion of the system. This is the principle of conservation of linear momentum.

NOTES

Total linear momentum is conserved in

(*i*) all processes of collision between particles,

(*ii*) sudden splitting of a body into fragments such as in the bursting of a bomb.

A quantity of interest in the motion of a particle or a system of particles is the angular momentum. For the system that we have considered above, the angular momentum of the k^{th} particle about the origin of coordinates is

$$\vec{M}_k = \vec{r}_k \times \vec{p}_k = \vec{r}_k \times m_k \dot{\vec{r}}_k \tag{8}$$

where m_k is the mass of the kth particle, $\vec{p}_k = m_k \dot{\vec{r}}_k$ is the linear momentum vector of the kth particle, $\dot{\vec{r}}_k$ being the velocity vector of the kth particle.

Using Equation (5) we have

$$\vec{r}_k \times \vec{F}_k = \vec{r}_k \times \frac{d}{dt}\left(\vec{p}_k\right) = \frac{d}{dt}\left(\vec{r}_k \times \vec{p}_k\right) \left(\text{because } \frac{d\,\vec{r}_k}{dt} \times \vec{p}_k = 0\right)$$

or, $$\vec{r}_k \times \vec{F}_k = \frac{d}{dt}\vec{M}_k \qquad \text{(Using Equation 8)}$$

If $\vec{\tau}_k$ be the torque of the force $\vec{F}_k$ about the origin of coordinates we have by definition

$$\vec{\tau}_k = \vec{r}_k \times \vec{F}_k$$

In view of the above two equations, we obtain

$$\vec{\tau}_k = \frac{d\,\vec{M}_k}{dt} \tag{9}$$

Summing over all the particles in the system, we get

$$\sum_{k=1}^{N} \vec{\tau}_k = \sum_{k=1}^{N} \frac{d}{dt}\vec{M}_k = \frac{d}{dt}\left(\sum_{k=1}^{N} \vec{M}_k\right)$$

or $$\vec{\tau} = \frac{d\,\vec{M}}{dt} \tag{10}$$

In the above $$\vec{\tau} = \sum_{k=1}^{N} \vec{\tau}_k$$

is the net torque of the forces acting on the particles of the system about the origin, and $\vec{M} = \sum_{k=1}^{N} \vec{M}_k$ is the total angular momentum of the system about the origin.

Equation (10) shows that if a system is devoid of any external torque about a point then the total angular momentum of the system about that point is conserved.

NOTES

In order to measure the total effect of a force it is usual to consider its line integral taken over the path of the particle from which it acts. From Equation (5) we get

$$\int_1^2 \vec{F}_k \cdot d\vec{r}_k = \int_1^2 \frac{d}{dt}\left(\vec{p}_k\right)\cdot d\vec{r}_k$$

$$= \int_1^2 \frac{d}{dt}\left(m_k \dot{\vec{r}}_k\right)\cdot d\vec{r}_k$$

$$= \left\{\frac{1}{2} m_k \dot{\vec{r}}_k^{\,2}\right\}_1^2 \qquad (11)$$

The above line integral is a scalar quantity and in physics it is referred to as the work done by the force when the particle undergoes a change of configuration from 1 to 2. In order to consider its effects in terms of conserved quantities it is assumed that the above work is stored up in the particle as the kinetic energy of motion. The quantity $\frac{1}{2} m_k \dot{\vec{r}}_k^{\,2}$ on the right hand side of Equation (11) is thus defined as the kinetic energy T_k of the particle k.

Summing over all the particles of the system, we get from Equation (11)

$$\int_1^2 \sum_{k=1}^{N} \vec{F}_k \cdot d\vec{r}_k = \sum_{k=1}^{N} \int_1^2 \vec{F}_k \cdot d\vec{r}_k = \sum_k \left. T_k \right|_1^2 = T^{(2)} - T^{(1)} \qquad (12)$$

where $T^{(1)}$ and $T^{(2)}$ are the initial and final values of the total kinetic energy of the system.

In several systems $\sum_{k=1}^{N} \vec{F}_k \cdot d\vec{r}_k$ may be expressed as a perfect differential $-dV$, where V is a function of the position coordinates of the particles in the system. Clearly, work done by the forces becomes independent of the actual paths followed by the particles and instead depends only upon their initial and final positions. Such systems are said to be conservative.

Thus, for conservative systems we get according to Equation (12)

$$V^{(1)} + T^{(1)} = V^{(2)} + T^{(2)} \qquad (13)$$

Equation (13) tells us that total energy $E = T + V$ of a conservative system is a constant or the total energy conserved though it may be exchanged between kinetic and potential energies.

We may note that there exists reality to kinetic energy but the same is not true for potential energy. Potential energy in some sense is a fictitious quantity. It is defined such that any change in its value is exactly compensated by the changes in the kinetic energy.

NOTES

Energy concept is of fundamental importance for mechanical systems because all the mechanical properties of complex systems can be understood by specifying the mathematical form of a limited number of scalar energy functions. Analytical mechanics consists of a general development of the above idea.

The conservation laws play an important role in mechanics. In some cases the identification of conserved quantities may be regarded as the solution to a problem.

Check Your Progress

1. Define a particle.
2. Express Newton's second law.

1.3 KEPLER'S LAWS OF PLANETARY MOTION

Based on the observations made by Tycho Brahe, Kepler enunciated the following three laws for the motion of planets round the sun.

1st Law: Each planet moves in an elliptical path with the sun at one of the foci of the ellipse.

2nd Law: The area swept by the radius vector (the line joining the sun to the planet) in equal intervals of time is equal, i.e., the areal velocity of the planet is a constant.

3rd Law: The square of the time period of revolution of the planet round the sun is directly proportional to the cube of the semi-major axis of the ellipse.

Derivation of Kepler's Laws

Consider a planet of mass M_p moving under the gravitational attraction of the sun of mass M_s. The force on the planet towards the sun when its distance from the sun is r is given by

$$F(r) = -G\frac{M_s M_p}{r^2} = -\frac{k}{r^2} \tag{14}$$

where $$k = GM_sM_p = \text{a constant} \tag{15}$$

The potential energy $U(r)$ corresponding to the force $F(r)$ is

$$U(r) = -\int F(r)\,dr = k\int\frac{dr}{r^2} = -\frac{k}{r} \tag{16}$$

In terms of the variable u, introduced in the previous section, we may write Equation (16) as

$$U\left(\frac{1}{u}\right) = -ku \tag{17}$$

NOTES

Substituting Equation (17) in the differential equation of motion given by Equation,

$$\frac{d^2u}{d\theta^2}+u = -\frac{m}{M^2}\frac{d}{du}U\left(\frac{1}{u}\right)$$

we get

$$\frac{d^2u}{d\theta^2}+u = -\frac{m}{M^2}\frac{d}{du}\ -ku$$

or
$$\frac{d^2u}{d\theta^2}+u = \frac{km}{M^2} \qquad (18)$$

The most general solution of the Equation (18) is

$$u = \frac{mk}{M^2}+u_o\cos\ \theta-\theta_o \qquad (19)$$

where u_o and θ_0 are constants. By orienting the coordinate system properly, let us, for convenience, choose the constant θ_0 equal to zero so that the Equation (19) takes the form

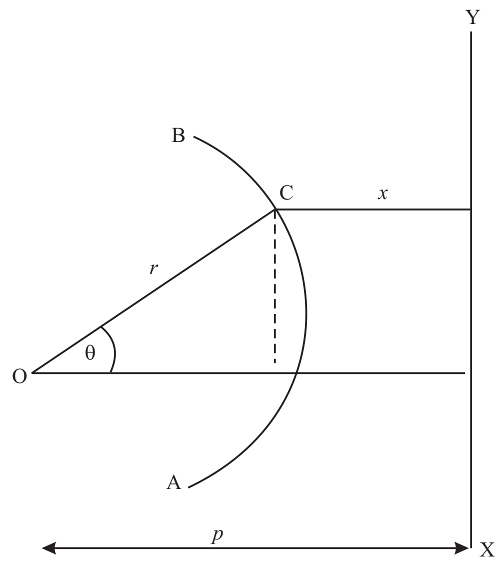

Fig. 1.1 *Conic Section*

$$\frac{1}{r} = \frac{km}{M^2}+u_o\cos\theta$$

or
$$r = \frac{1}{\dfrac{mk}{M^2}+u_o\cos\theta}$$

or
$$r = \frac{\dfrac{M^2}{mk}}{1+\dfrac{u_oM^2}{mk}\cos\theta} \qquad (20)$$

NOTES

We find that for $\theta = 0$, r is the maximum, while for $\theta = 0 + \pi$, r is the minimum. We can thus interpret θ_0, which we have taken as zero, to represent one of the angles corresponding to a turning point in the path of motion.

Equation (20) can be compared with the equation of a conic section which is a curve AB as shown in the Figure 1.1. In the figure, O is a fixed point called the focus and XY a fixed line called the directrix of the conic section. Let C be any arbitrary point on the curve AB.

$$\text{OC} = r(\text{say})$$

Let CD be the perpendicular from the point C on the directrix.

$$\text{CD} = x(\text{say})$$

Let the line OC make an angle θ with the line drawn normal from the focus to the directrix.

The curve AB is such that the ratio $\frac{r}{x}$ is a constant. This constant ratio is called the eccentricity of the conic section and is usually denoted by the symbol ε.

Let p be the distance of the directrix from the focus. We then get according to the Figure 1.1.

$$p = x + r\cos\theta = \frac{r}{\varepsilon} + r\cos\theta$$

Semilatus rectum, which we denote by the symbol ρ of the conic section is defined as

$$\rho = \varepsilon p \text{ (= constant)}$$

The above gives $$p = \frac{\rho}{\varepsilon}$$

Thus, we get $$\frac{\rho}{\varepsilon} = \frac{r}{\varepsilon} + r\cos\theta = \frac{r}{\varepsilon}[1 + \varepsilon\cos\theta]$$

or $$r = \frac{\rho}{1 + \varepsilon\cos\theta} \tag{21}$$

Equation (21) is the general equation of a conic section. Comparing Equation (20) with Equation (21), we find that if motion takes place under a central attractive force varying inversely as the square of the distance from the force centre then the path is a conic section having the focus at the force centre; the eccentricity and the semilatus rectum of the conic section being given by

$$\varepsilon = \frac{u_o M^2}{mk} \tag{22}$$

$$\rho = \frac{M^2}{mk} \tag{23}$$

NOTES

To know exactly the eccentricity of the conic section in which the motion takes place we are required to find the constant u_o in terms of known quantities.

For the motion which is under consideration, the total energy is given by

$$E = \frac{1}{2}m\dot{r}^2 + \frac{M^2}{2mr^2} - \frac{k}{r} = \text{constant} \tag{24}$$

At the turning point corresponding to $r = r_{\min}$, we get according to Equation (24)

$$E = \frac{M^2}{2mr_{\min}^2} - \frac{k}{r_{\min}} \tag{25}$$

From Equation (21) we have

$$r_{\min} = \frac{\rho}{1+\varepsilon} = \frac{M^2}{mk(1+\varepsilon)} \tag{26}$$

Using Equation (26) in Equation (25) we obtain

$$E = \frac{M^2m^2k^2(1+\varepsilon)^2}{2mM^4} - \frac{kmk(1+\varepsilon)}{M^2}$$

or $$E = \frac{1}{2}mk^2\frac{(1+\varepsilon)^2}{M^2} - \frac{mk^2}{M^2}(1+\varepsilon)$$

or $$E = \frac{mk^2}{2M^2}\left[1+\varepsilon^2+2\varepsilon-2-2\varepsilon\right] = \frac{mk^2}{2M^2}(\varepsilon^2-1)$$

or $$\varepsilon = \left[1+\frac{2M^2E}{mk^2}\right]^{\frac{1}{2}} \tag{27}$$

Substituting for ε given by Equation (27), the equation for the conic section in which the motion takes place is given by

$$r = \frac{\frac{M^2}{mk}}{1+\left[1+\frac{2M^2E}{mk^2}\right]^{\frac{1}{2}}\cos\theta} \tag{28}$$

From Equation (27) we find that the eccentricity ε and hence the nature of the conic section is primarily decided by the total energy E. We the get

(*i*) For $E > 0$, i.e., the total energy being positive, the eccentricity is greater than 1 and the conic section is a hyperbola,

(*ii*) For $E = 0$, the eccentricity is 1 and the conic section is a parabola, (*iii*) For $E < 0$, i.e., the total energy being negative, the eccentricity is less than 1 and the conic section is an ellipse,

(*iv*) For eccentricity equal to 0, the conic section is a circle.

NOTES

For the motion of the planet in the gravitational field of the sun which is being considered presently we have the following:

Kinetic energy and potential energy of the planet when it is at a distance r from the sun and has velocity v are

$$T = \frac{1}{2} M_p v^2$$

and
$$U = -G \frac{M_s M_p}{r^2}$$

Clearly, the total energy of the planet is

$$E = T + U = \frac{1}{2} M_p v^2 - G \frac{M_s M_p}{r} \quad (29)$$

The necessary centripetal force for the planet to move along the conic is provided by the gravitational force of attraction on the planet due to the sun. Thus, we have

$$\frac{M_p v^2}{r} = G \frac{M_s M_p}{r^2}$$

or
$$M_p v^2 = G \frac{M_s M_p}{r} \quad (30)$$

Using Equation (30) in Equation (29), we get

$$E = \frac{1}{2} G \frac{M_s M_p}{r} - G \frac{M_s M_p}{r} = -\frac{1}{2} G \frac{M_s M_p}{r} \quad (31)$$

We find the total energy E of the planet to be negative. Clearly, the eccentricity of the conic section is less than 1, and consequently the planet goes round in an elliptic path with the sun at one of its foci. This is Kepler's first law.

We have seen that in the case of motion under central force, the angular momentum is a constant of the motion Equation

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2 \dot{\theta} = \text{constant} = M(\text{say})$$

$$mr^2 \dot{\theta} = \text{constant} = M \quad (32)$$

NOTES

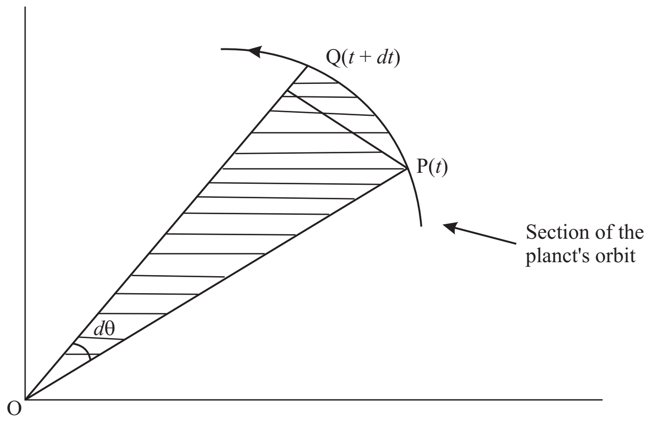

Fig. 1.2 *Positions of the Planet*

Consider Figure 1.2, in which the positions of the planet on its path of motion at two instants of time t and $t + dt$ are shown. During the interval dt the area swept dA by the radius vector is the area of the shaded region. Since dt is infinitesimally small, the arc PQ can be considered as a straight line. Hence, we get

$$dA = \text{Area of the triangle } OPQ$$

$$= \frac{1}{2} r r d\theta = \frac{1}{2} r^2 d\theta$$

Thus, the areal velocity of the planet is

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

In view of Equation (32), the above becomes

$$\frac{dA}{dt} = \frac{1}{2} \frac{M}{m} = \text{constant} \tag{33}$$

The above is the Kepler's second law of planetary motion.

Kepler's third law can be proved as follows:

If a be the semi-major axis of the ellipse in which the planet moves, we get by definition

$$a = \frac{\rho}{1 - \varepsilon^2} \tag{34}$$

Substituting for ρ given by Equation (23), we get

$$a = \frac{M^2}{mk\left(1 - \varepsilon^2\right)}$$

or
$$1-\varepsilon^2 = \frac{M^2}{mka} \tag{35}$$

NOTES

Using Equation (35) in the expression for E given by Equation (27), we get

$$E = -\frac{mk^2}{2M^2}\frac{M^2}{mka} = -\frac{k}{2a} \tag{36}$$

The semi-minor axis b of the ellipse is related to a and ε as

$$b = a\left(1-\varepsilon^2\right)^{\frac{1}{2}} \tag{37}$$

Using Equation (35) the above becomes

$$b = a\frac{M}{(mka)^{\frac{1}{2}}} = \frac{Ma^{\frac{1}{2}}}{(mk)^{\frac{1}{2}}} \tag{38}$$

Let T be the time period of revolution of the planet in its elliptic orbit. We then have

$$\text{Area of the ellipse} = \int_0^T \left(\frac{dA}{dt}\right)dt$$

or
$$\pi ab = \frac{M}{2m}T$$

or
$$T = \frac{2m}{M}\pi ab$$

Substituting for a and b in the above, we get

$$T = \frac{2m}{M}\pi a\frac{Ma^{\frac{1}{2}}}{(mk)^{\frac{1}{2}}} = \frac{2\pi ma^{\frac{3}{2}}}{(mk)^{\frac{1}{2}}}$$

Substituting for k, the above gives

or
$$T^2 = \frac{4\pi^2}{\left(M_p+M_s\right)GM_sM_p}a^3$$

or
$$T^2 = \frac{4\pi^2}{G\left(M_s+M_p\right)}a^3 \tag{39}$$

Thus, we find that

$$T^2 \propto a^3 \tag{40}$$

The above is the Kepler's third law of planetary motion.

NOTES

Check Your Progress

3. Define Kepler's first law of planetary motion.
4. State Kepler's second law of planetary motion.
5. What is meant by Kepler's third law of planetary motion?

1.4 STABILITY OF ORBIT

General Description of the Stability of Planetary Orbits

First, we briefly look at the general character of the long-term stability of planetary orbits. Our interest here focuses particularly on the inner four terrestrial planets for which the orbital time-scales are much shorter than those of the outer five planets. As we can see clearly from the planar orbital configurations shown in Figures 1.3 and 1.4, orbital positions of the terrestrial planets differ little between the initial and final part of each numerical integration, which spans several Gyr. Here Gyr refers to Billion years. A billion years (109 years) is a unit of time on the petasecond scale, more precisely equal to 3.16×1016 seconds. The solid lines denoting the present orbits of the planets lie almost within the swarm of dots even in the final part of integrations (b) and (d). This indicates that throughout the entire integration period the almost regular variations of planetary orbital motion remain nearly the same as they are at present.

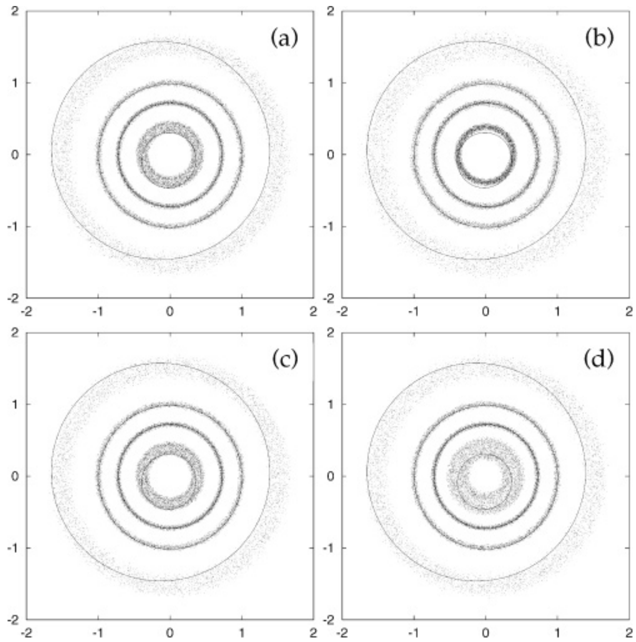

Fig. 1.3 *Vertical View of the Four Inner Planetary Orbits for* $N_{\pm 1}$

NOTES

Figure 1.3 illustrates the vertical view of the four inner planetary orbits (from the z -axis direction) at the initial and final parts of the integrations $N_{\pm1}$. The axes units are au. The x- and y- plane is set to the invariant plane of Solar system total angular momentum.

(a) The initial part of N_{+1} ($t = 0$ to $0.0547 \times 10\ 9$ yr).

(b) The final part of N_{+1} ($t = 4.9339 \times 10\ 8$ to $4.9886 \times 10\ 9$ yr).

(c) The initial part of N_{-1} ($t = 0$ to $-\ 0.0547 \times 109$ yr) .

(d) The final part of N_{-1} ($t = -3.9180 \times 10\ 9$ to $-3.9727 \times 10\ 9$ yr).

In each panel, a total of 23 684 points are plotted with an interval of about 2190 yr over 5.47×10^7 yr. Solid lines in each panel denote the present orbits of the four terrestrial planets.

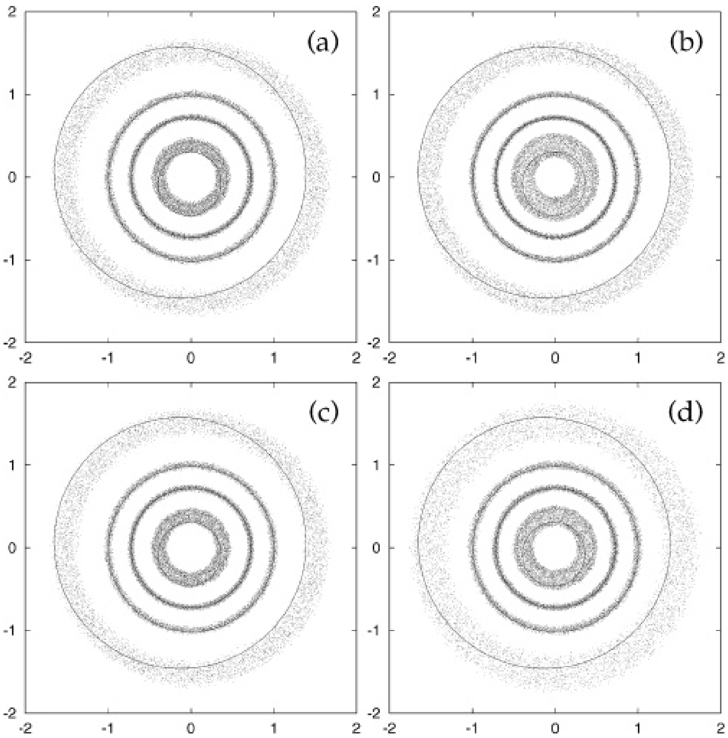

***Fig. 1.4** Vertical View of the Four Inner Planetary Orbits for $N_{\pm 2}$*

Same as in Figure 1.3, but for $N_{\pm 2}$.

(a) The initial part of N_{+2} ($t = 0$ to 0.0547×10^{9} yr).

(b) The final part of N_{+2} ($t = 4.9829 \times 10^{8}$ to 5.0376×10^{9} yr).

(c) The initial part of N_{-2} ($t = 0$ to $-\ 0.0547 \times 10^{9}$ yr).

(d) The final part of N_{-2} ($t = -\ 3.9726 \times 10^{9}$ to $-\ 3.9179 \times 10^{9}$ yr).

The variation of eccentricities and orbital inclinations for the inner four planets in the initial and final part of the integration N_{+1} is shown in Figure 1.5. As expected, the character of the variation of planetary orbital elements does

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not differ significantly between the initial and final part of each integration, at least for Venus, Earth and Mars. The elements of Mercury, especially its eccentricity, seem to change to a significant extent. This is partly because the orbital time-scale of the planet is the shortest of all the planets, which leads to a more rapid orbital evolution than other planets; the innermost planet may be nearest to instability. This result appears to be in some agreement with Laskar's expectations that large and irregular variations appear in the eccentricities and inclinations of Mercury on a time-scale of several 10^9 yr. However, the effect of the possible instability of the orbit of Mercury may not fatally affect the global stability of the whole planetary system owing to the small mass of Mercury.

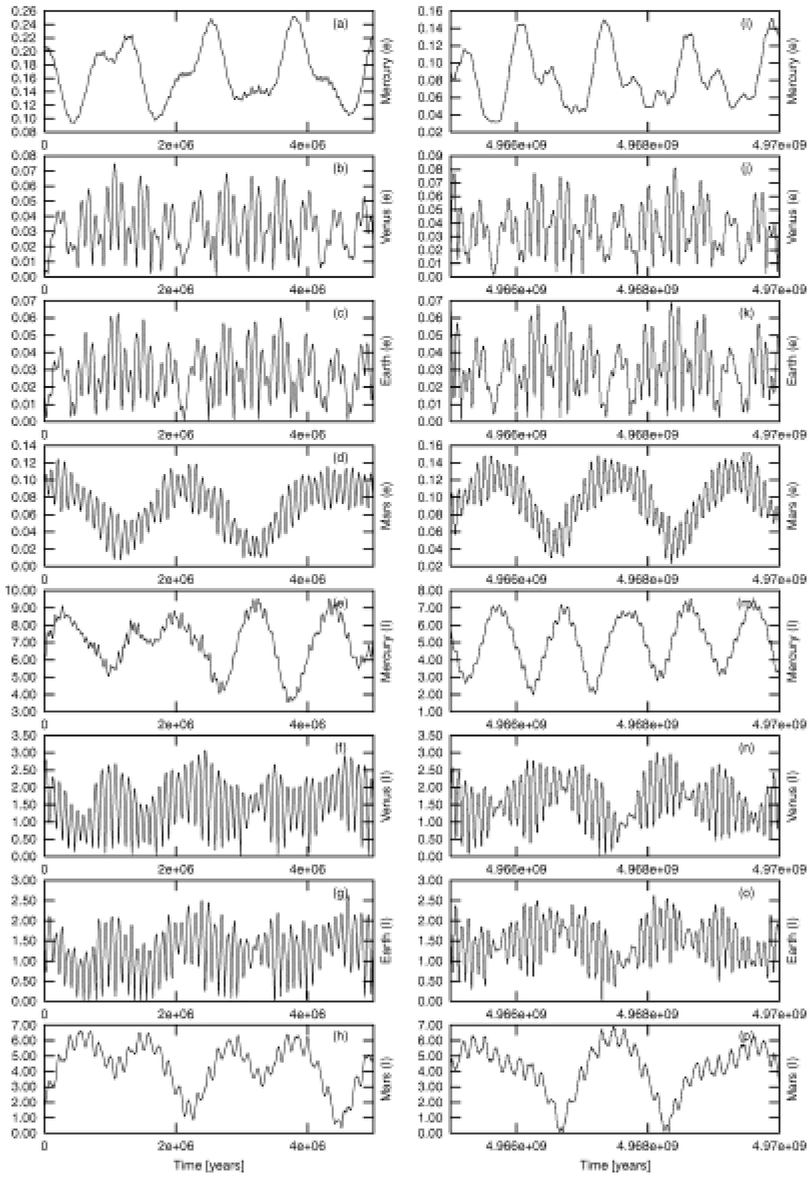

Fig. 1.5 *Variation of Eccentricities and Orbital Inclinations*

Figure 1.5 illustrates the eccentricities and inclinations of the four inner planetary orbits in the initial and final parts of the integration N_{+1}. Where,

(a)–(d) Eccentricities of Mercury, Venus, Earth and Mars at the beginning of the integration.

(e)–(h) Inclinations of Mercury, Venus, Earth and Mars at the beginning of the integration.

(i)–(l) Eccentricities of Mercury, Venus, Earth and Mars at the end of the integration.

(m)–(p) Inclinations of Mercury, Venus, Earth and Mars at the end of the integration.

All the elements are reckoned on the Solar system invariable plane with a heliocentric origin. The unit of inclination is the degree. The orbital motion of the outer planets seems rigorously stable and quite regular over this time-span.

1.5 CLASSIFICATION OF DYNAMICAL SYSTEM

In mathematics, a dynamical system is a system in which a function describes the time dependence of a point in a geometrical space. Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe, and the number of fish each springtime in a lake, and so on.

At any given time, a dynamical system has a state given by a tuple of real numbers (a vector) that can be represented by a point in an appropriate state space (a geometrical manifold). The evolution rule of the dynamical system is a function that describes what future states follow from the current state. Often the function is deterministic, that is, for a given time interval only one future state follows from the current state. However, some systems are stochastic, in that random events also affect the evolution of the state variables.

In physics, a dynamical system is described as a "particle or ensemble of particles whose state varies over time and thus obeys differential equations involving time derivatives." The dynamical system define the key theoretical concepts of phase space and fixed points (or limit cycles). Fundamentally, the concept of a dynamical system has its origins in Newtonian mechanics. There, as in other natural sciences and engineering disciplines, the evolution rule of dynamical systems is an implicit relation that gives the state of the system for only a short time into the future. The relation is either a differential equation, difference equation or other time scale. Following are the standard classifications of dynamical systems.

1.5.1 Dynamical Systems of Order *n*

NOTES

A dynamical system of order n is defined as follows:

1. The state of the system at any time t is represented by n-real variables as coordinates of a vector $\vec{r}$ in an abstract n- dimensional space as,

$$\{x_1, x_2, x_3 \cdots, x_n\} \Rightarrow \vec{r}$$

   We refer to this space as the state space of simply phase space in keeping with its usage in Hamiltonian dynamics. Thus the state of the system at any given time is a point in this phase space.

2. In the time evolution of the system, motion is represented by a set of first order equations also called as the 'equations of motion' and is expressed as,

$$\frac{dx_1}{dt} = v_1(x_1, x_2, \cdots, x_n, t)$$
$$\frac{dx_2}{dt} = v_2(x_1, x_2, \cdots, x_n, t)$$
$$\cdots$$
$$\frac{dx_n}{dt} = v_n(x_1, x_2, \cdots, x_n, t) \qquad (41)$$

   Or simply,

$$\frac{d\vec{r}}{dt} = \vec{v}(\vec{r}, t)$$

   Where,

$$\vec{v} \Rightarrow \{v_1, v_2, \cdots, v_n\}$$

This is called the *velocity function*. While we use the notation x and v in analogy with mechanics, they do not always have the usual meaning of position and velocity.

The above mentioned Statements 1 and 2 together define a dynamical system of order n. If the velocity function does not depend on time explicitly, then the system is time independent or autonomous. The set of all possible motions in called phase flow. The name phase flow may be understood by imagining a fluid flowing in the phase space with velocity $\vec{v}$. The direction and magnitude of the flow at any point in phase space is determined by the velocity vector.

For the solution to be unique requires $\vec{v}$ to obey certain conditions. Without going to mathematical details, it suffices to say that the solution of the differential Equations (41) are unique if the velocity vector $\vec{v}$ is a continuous function of its arguments and at least once differentiable. With the time evolution, the initial state of the system (denoted by a point in the phase space) evolves and follows a continuous trajectory which we shall call a phase curve which may be closed or open. Distinct phase curves are obtained when the initial state of the system is specified by a point which is not one of the points on the other trajectory. This leads to an important fact that two distinct trajectories cannot intersect in a finite time period.

1.5.2 First Order Systems

This is the simplest case of a dynamical system. The equation of motion is given by,

$$\frac{dx}{dt} = v(x, t)$$

Where v is the velocity function. For any given $v(x, t)$, $x(t)$ is completely determined given $x(t)$ at some $t = t_0$. If, in particular, the system is autonomous, or v is not explicitly dependent on time then the solution can be written as,

$$t - t_0 = \int_{x(t_0)}^{x(t)} \frac{dx'}{v(x')}$$

Thus the solution $x(t)$ depends only on the difference $(t = t_0)$. Thus the time evolution of the system depends entirely on the time elapsed no matter where the origin of time is fixed.

1.5.3 Second Order System

The phase space is two dimensional and each point in the phase space is characterized by two real numbers (x, y).

$$\vec{r}(t) \Rightarrow (x(t), y(t))$$

Dynamical evolution of the system is governed by the system of equations,

$$\frac{dx}{dt} = v_x(x, y, t)$$

$$\frac{dy}{dt} = v_y(x, y, t)$$

Or simply,

$$\frac{d\vec{r}}{dt} = \vec{v}(x, y, t)$$

The solution of the equations defined by the velocity vector $\vec{v}$ has a unique solution for all time with the initial condition $\vec{r}(t_0) = (x_0, y_0)$. If the system is autonomous then of course there is no explicit dependence on time in the velocity function.

NOTES

The solution $\vec{r}(t)$ obtained with a particular initial condition defines a continuous curve called the phase curve. The set of all phase curves tracing the actual motion is called phase flow. Note the phase curves exist only when $d \geq 2$. For $d = 1$there are only phase flows. The equation of the phase curve for an autonomous system of order 2 is given by,

$$\frac{dy}{dx} = \frac{v_y(x, y)}{v_x(x, y)}$$

Check Your Progress

6. Define a dynamical system and give its examples.
7. What is phase flow?

1.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. A particle is defined as a body whose position in space can be exactly defined by three coordinates which may be cartesian, spherical polar, cylindrical, etc. This is an idealization and does not apply to any real body.
2. Considering the concept of a mass point, the law of motion, namely Newton's second law for a body, can be written as

$$m\frac{d^2 \vec{r}}{dt^2} = \vec{F}$$

   In the above, $\vec{F}$ is the resultant of all the forces applied to the body and, $\frac{d^2 \vec{r}}{dt^2}$ is the acceleration vector. The quantity m characterizes the mass point and is called its mass.
3. Each planet moves in an elliptical path with the sun at one of the foci of the ellipse.
4. The area swept by the radius vector (the line joining the sun to the planet) in equal intervals of time is equal, i.e., the areal velocity of the planet is a constant.
5. The square of the time period of revolution of the planet round the sun is directly proportional to the cube of the semi-major axis of the ellipse.
6. In physics, a dynamical system is described as a "particle or ensemble of particles whose state varies over time and thus obeys differential

equations involving time derivatives." Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe, and the number of fish each springtime in a lake, and so on.

7. The set of all phase curves tracing the actual motion is called phase flow.

NOTES

1.7 SUMMARY

- If a system of particles is devoid of any net external force then the total linear momentum (magnitude as well as direction) of the system remains conserved throughout the motion of the system.
- if motion takes place under a central attractive force varying inversely as the square of the distance from the force centre then the path is a conic section having the focus at the force centre; the eccentricity and the semilatus rectum of the conic section being given by

$$\varepsilon = \frac{u_o M^2}{mk} \quad \text{and} \quad \rho = \frac{M^2}{mk}$$

- Kinetic energy and potential energy of the planet when it is at a distance r from the sun and has velocity v are

$$T = \frac{1}{2} M_p v^2 \quad \text{and} \quad U = -G\frac{M_s M_p}{r^2}$$

- All the elements are reckoned on the Solar system invariable plane with a heliocentric origin.
- A dynamical system is a system in which a function describes the time dependence of a point in a geometrical space
- At any given time, a dynamical system has a state given by a tuple of real numbers (a vector) that can be represented by a point in an appropriate state space (a geometrical manifold).
- Two distinct trajectories cannot intersect in a finite time period.

1.8 KEY WORDS

- **Linear momentum:** It is the product of the mass and velocity of an object. It is a vector quantity, possessing a magnitude and a direction in three-dimensional space.
- **Mass:** It is a measure of the amount of matter in an object. Mass measures the quantity of matter regardless of both its location in the universe and the gravitational force applied to it. An object›s mass is constant in all circumstances.

- **Dynamical system:** It is a system in which a function describes the time dependence of a point in a geometrical space.
- **Phase flow:** The set of all phase curves tracing the actual motion is known as phase flow.

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1.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Write short notes on the followings:
   (i) Frames of reference
   (ii) Mass point or mass
2. Derive Kepler's first law of planetary motion.
3. Give a general description of stability of planetary orbits.
4. Write a short note on dynamical systems of order n.

Long-Answer Questions

1. Give a detailed account of Newton's laws of motion.
2. Discuss Kepler's laws of planetary motion.
3. Describe first and second order dynamical systems.

1.10 FURTHER READINGS

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UNIT 2 LAGRANGE EQUATION

NOTES

Structure

2.0 INTRODUCTION

Swiss mathematician Leonhard Euler and Italian-French mathematician Joseph Louis Lagrange evolved Lagrange's equation in relationship with their discussion of the tautochrone problem. They used Lagrange's method to mechanics, which led to the formation of Lagrangian mechanics. Lagrange's equation is a second-order partial differential equation whose solutions are the functions for which a given functional is stationary. In this unit you will study Lagrange's equations for simple systems and important properties of the Lagrangian function. You will learn principle of virtual work and D'Alembert's principle. You will derive Lagrange's equation for general and conservative system. Application of Lagrangian formulation is also discussed in detail.

2.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain Lagrange's equations for simple systems
- Define gauge function, law of inertia, central force and virtual displacement
- Understand D' Alembert's principle and principle of virtual work
- Derive Lagrange's equations for general and conservative systems

- Discuss applications of Lagrangian formulation
- Describe important properties of the Lagrangian function

NOTES

2.2 LAGRANGE'S EQUATIONS FOR SIMPLE SYSTEMS

Using Hamilton's principle of least action, it is possible to derive the dynamical equations for the system under consideration.

We have the Lagrangian of the system given by

$$L = L(q_1, \ldots.., q_s, \dot{q}_1, \ldots.., \dot{q}_s, t)$$

For convenience, we may write L in a shorter form as

$$L = L(q_k, \dot{q}_k, t) \tag{1}$$

In the above, q_k stands for all the coordinates and $\dot{q}_k$ stands for all the velocities which describe the system.

q_k's in general depend on time explicitly so that we should write $q_k(t)$ instead of q_k. Let $q_k(t)$ be replaced by $q_k(t) + \delta q_k(t)$. where $\delta q_k(t)$ is a small variation in $q_k(t)$ in the interval of time from time t_1 to time t_2. The variation of action S for fixed t_1 and t_2 is then

$$\delta S = \int_{t_1}^{t_2} L(q_k + \delta q_k, \dot{q}_k + \delta\dot{q}_k, t)\, dt - \int_{t_1}^{t_2} L(q_k, \dot{q}_k, t) \tag{2}$$

The major contributions in the expansion of $L\ (q_k + \delta q_k, \dot{q}_k + \delta\dot{q}_k, t)$ in powers of δq_k and $\delta\dot{q}_k$ are of the first order. Hence, for S to be an extremum, these terms should be zero. Thus, Hamilton's principle given by the following equation:

$$\delta S = \delta\int_{t_1}^{t_2} L(q_1, \ldots.., q_s, \dot{q}_1, \ldots.., \dot{q}_s, t) = 0$$

takes the form,

$$\delta S = \delta\int_{t_1}^{t_2} L(q_k, \dot{q}_k, t)\, dt = \int_{t_1}^{t_2} \sum_{k=1}^{s} \left(\frac{\partial L}{\partial q_k}\delta q_k + \frac{\partial L}{\partial \dot{q}_k}\delta\dot{q}_k \right) dt = 0 \tag{3}$$

We may note that t is fixed in the δ-variation under consideration. The following identity holds

$$\delta\dot{q}_k = \frac{d}{dt}(\delta q_k) \tag{4}$$

Using Equation (4) in Equation (3) we get

$$\delta S \quad \int_{t_1}^{t_2} \sum_{k=1}^{s} \left(\frac{\partial L}{\partial q_k}\delta q_k \right) dt + \int_{t_1}^{t_2} \sum_{k=1}^{s} \left(\frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}(\delta\dot{q}_k) \right) dt = 0$$

NOTES

or $$\delta S = \int_{t_1}^{t_2} \sum_k \left(\frac{\partial L}{\partial q_k} \delta q_k \right) dt + \sum_{k=1}^{s} \left\{ \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \right\}_{t_1}^{t_2} - \sum_{k=1}^{s} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \right] dt = 0 \qquad (5)$$

Since variations at the end points are zero, i.e.,

$$\delta q_k(t_1) = 0 = \delta q_k(t_2) \qquad (6)$$

Equation (5) becomes

$$\delta S = \int_{t_1}^{t_2} \sum_{k=1}^{s} \left(\frac{\partial L}{\partial q_k} \delta q_k \right) dt - \int_{t_1}^{t_2} \sum_{k=1}^{s} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \delta q_k \, dt = 0$$

or $$\delta S = \sum_{k=1}^{s} \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) \right] \delta q_k \, dt = 0 \qquad (7)$$

The result given by Equation (7) holds for all arbitrary variations provided the coefficient of δq_k in the integrand on the right hand side vanishes for each k. We thus obtain

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = 0$$

or $$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0; \quad (k = 1, 2, ..., s) \qquad (8)$$

The above set of s number of second order differential equations satisfied by the Lagrangian of the system are called the Lagrange's equations of motion.

Lagrange's equations of motion given by Equation (8) can be seen to follow directly from Euler-Lagrange equation given by Equation

$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'_k} \right) = \frac{\partial f}{\partial y_k}; \qquad k = 1, ..., s$$

. If in the function f given by Equation ($f = f(y_1, ..., y_s, y'_1, ..., y'_s, x)$ we replace y_1, ….., y_s by the generalized coordinates q_1, ….., q_s, respectively, y_1', ….., y_s' by the generalized velocities $\dot{q}_1, ..., \dot{q}_s$ respectively and x by t then the function f can be identified as the Lagrangian $L(q_1, ..., q_s, \dot{q}_1, ..., \dot{q}_s, t)$ and the Euler-Lagrange equations given by Equation

$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'_k} \right) = \frac{\partial f}{\partial y_k}; \qquad k = 1, ..., s$$

become the Lagrange's equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = \frac{\partial L}{\partial q_k}; \quad k = 1, ….., s.$$

NOTES

2.2.1 Important Properties of the Lagrangian Function

In this section we discuss the important properties of the Lagrangian function and attempt to find its meaning, i.e., to find whether or not the Lagrangian function is representative of some physical quantity of the system under consideration.

(i) Lagrangian is Gauge Invariant

The Lagrangian function of a system having s degrees of freedom and described by the generalized coordinates q_1, ….., q_s and the generalized velocities $\dot{q}_1, ..., \dot{q}_s$ is given by

$$L = L(q_1, ..., q_s, \dot{q}_1, ..., \dot{q}_s, t) = L(q_k, \dot{q}_k, t) \qquad (9)$$

Consider an arbitrary function $F = F(q_1, ..., q_s, t) = F(q_k, t)$ and define a new function $L'(q_k, \dot{q}_k, t)$ as

$$L'(q_k, \dot{q}_k, t) = L(q_k, \dot{q}_k, t) + \frac{d}{dt}F(q_k, t) \qquad (10)$$

The action of the system between the time limits t_1 and t_2 is the time integral

$$S = \int_{t_1}^{t_2} L(q_k, \dot{q}_k, t)\,dt \qquad (11)$$

Let us consider the time integral of the new function L' between the same time limits

$$S' = \int_{t_1}^{t_2} L'(q_k, \dot{q}_k, t)\,dt \qquad (12)$$

Using Equation (10) in Equation (12) we get

$$S' = \int_{t_1}^{t_2} \left[L(q_k, \dot{q}_k, t) + \frac{d}{dt}F(q_k, t) \right] dt$$

$$= \int_{t_1}^{t_2} L(q_k, \dot{q}_k, t)\,dt + \int_{t_1}^{t_2} \frac{d}{dt}F(q_k, t)\,dt$$

or

$$S' = S + \{F(q_k, t)\}_{t_1}^{t_2} \qquad (13)$$

Taking δ-variation of Equation (13) we get

$$\delta S' = \delta S + \delta\{F(q_k, t)\}_{t_1}^{t_2}$$

or,

$$\delta S' - \delta S = \delta[F(q_k, t)]_{t=t_2} - \delta[F(q_k, t)]_{t=t_1} = 0 \qquad (14)$$

because $\delta q_k = 0$ at $t = t_1$ and at $t = t_2$.

According to Hamilton's principle we have

$$\delta S = 0$$

NOTES

In view of this and Equation (14) we find

$$\delta S' = 0 \tag{15}$$

The condition $\delta S' = 0$ leads to equations of motion which are the same as those given by the condition $\delta S = 0$. Hence, we may identify the function L' given by Equation (10) also as the Lagrangian of the system.

From the above we may conclude that the Lagrangian of a system cannot be defined uniquely, but can be defined only within an additive total time derivative of any function of coordinates relevant to the system and time.

The arbitrary function $F(q_1, ..., q_s, t) = F(q_k, t)$ is called **gauge function**. Hence, the above result shows that Lagrangian of a system is **gauge invariant**.

(ii) Lagrangian is Additive

Let A and B be two non-interacting parts of a mechanical system. Let L_A and L_B be their Lagrangians, respectively. By additivity, we mean that the Lagrangian of the whole system is given by

$$L = L_A + L_B \tag{16}$$

A consequence of this property is that the equation of motion of the part A are completely independent of the quantities of the part B and vice-versa.

(iii) Lagrangian of a System is Arbitrary within an Overall Multiplicative Constant

This property means that if the Lagrangian is multiplied by any arbitrary constant then the equations of motion remain unaltered.

2.2.2 Lagrangian of a Particle Moving Freely in Space

Consider a particle of mass m moving freely in space with respect to the origin of an inertial frame of reference. Let $\vec{r}$ be the position vector of the particle at the instant of time t. Let $\vec{v}$ be the velocity of the particle at the instant t as observed from the frame under consideration. The Lagrangian function of the particle is given by

$$L = L(\vec{r}, \vec{v}, t) \tag{17}$$

From the symmetry properties of space and time with respect to an inertial frame, namely, homogeneity and isotropy of space and homogeneity of time, the Lagrangian for the free particle must be invariant with respect to (*i*) Translation in space (*ii*) Rotation in space about any axis, and (*iii*) Translation in time.

In other words, L cannot be an explicit function of $\vec{r}$ and t. Further, L should not depend upon the direction of the velocity of the particle. Thus, L

can only be a function of the magnitude of the velocity of the particle. Thus, we may write

$$L = L(v^2) \tag{18}$$

NOTES

According to Equation (8), Lagrange's equation of motion for the system is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = \frac{\partial L}{\partial \vec{r}}$$

or,
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \vec{v}}\right) = 0 \quad \left(\because \frac{\partial L}{\partial \vec{r}} = 0\right)$$

or,
$$\frac{\partial L}{\partial \vec{v}} = \text{A constant of motion} \tag{19}$$

Since L is a function of only velocity, the above equation leads to

$$\vec{v} = \text{Constant of motion} \tag{20}$$

Equation (20) is the **law of inertia** according to which a particle which moves without the influence of any external agent has a constant velocity vector.

To find the exact form of the function $L(v^2)$, consider two inertial frames of reference S and S', where S' moves with an infinitesimal uniform velocity, say $\vec{\varepsilon}$, with respect to S. Let L and L' be the Lagrangians of the particle as observed from the frames S and S', respectively. Since the equations of motion remain the same in all inertial frames, we must have L and L', different by only a total time derivative of a function of coordinates and time. We have according to Equation (10)

$$L' = L\left(v'^2\right) = L\left(v^2 + 2\vec{v}\cdot\vec{\varepsilon} + \varepsilon^2\right)$$

$$= L\left(v^2\right) + 2\vec{v}\cdot\vec{\varepsilon}\frac{\partial L}{\partial v^2} + \text{terms with higher order of } \varepsilon$$

$$= L + 2\vec{v}\cdot\vec{\varepsilon}\frac{\partial L}{\partial v^2}$$

Thus,
$$L' - L = 2\vec{v}\cdot\vec{\varepsilon}\frac{\partial L}{\partial v^2} \tag{21}$$

Now for $L' - L$ to be a total time derivative of a function of coordinates and time, we must have $L' - L$ as a linear function of $\vec{v}$, i.e., $\frac{\partial L}{\partial v^2}$ to be independent of the velocity. Further, $\frac{\partial \vec{\varepsilon}}{\partial t} = 0$. We thus find that L must be proportional to v^2, i.e., $L \propto v^2$. To assign a physical meaning to L, we take the constant of proportionality as $\frac{1}{2}m$, so that we can write

$$L = \frac{1}{2}mv^2 = \text{Kinetic energy of the particle.} \quad (22)$$

In order to obtain the Lagrangian function for an assembly of particles (non-interacting or interacting) we need to consider the important properties which the Lagrangian of the system possesses.

Check Your Progress

1. What are the Lagrange's equations?
2. What is the equation for the Lagrangian function of a system?
3. What is gauge function?
4. Define law of inertia?

2.3 PRINCIPLE OF VIRTUAL WORK

The work of a force on a particle along a virtual displacement is known as the virtual work. The principle of virtual work explains that in equilibrium the virtual work of the forces applied to a system is zero.

2.3.1 Virtual Displacement

Consider a system of N particles 1, 2, ….., N having s degrees of freedom. Let $q_1, q_2, \ldots, q_s$ be the generalized coordinates that describe the system. The configuration space of the system is s-dimensional. At any instant of time t, the configuration of the system is specified by a point in the configuration space, the point being defined by a particular set of values for the generalized coordinates.

Let the system be subjected to arbitrary displacement in the configuration space consistent with the constraints imposed on the system at the instant. The corresponding change in the configuration of the system is independent of time, i.e., no actual displacement of the system occurs with respect to time. Such displacements in the configuration space are called **virtual displacements**. It is usual to denote virtual displacement of the generalized coordinates, say q_k, as δq_k.

The concept of virtual displacement has been found useful for mathematical analysis of the properties of mechanical systems.

NOTES

2.3.2 Virtual Work

Consider the system of N particles described above. Let $\vec{F}_1, \vec{F}_2, ..., \vec{F}_N$ be the forces acting on the particles. If we consider the system to be in equilibrium, we have

$$\vec{F}_k = 0 \quad (k = 1, 2, ..., N)$$

(23)

Let $\vec{r}_1, \vec{r}_2, ..., \vec{r}_N$ be the equilibrium position vectors of the particles.

Let $\delta\vec{r}_1, \delta\vec{r}_2, ..., \delta\vec{r}_N$ be the infinitesimal virtual displacements of the particles from their equilibrium positions.

We then have according to Equation (23)

$$\sum_{k=1}^{N} \vec{F}_k \cdot \delta\vec{r}_k = 0 \qquad (24)$$

However, if the forces $\vec{F}_k$ are continuous functions of positions then the left hand side of Equation (24) can be interpreted as the net work done in the virtual displacements of the particles. If the system changes from its equilibrium configuration, we may write Equation (24) as

$$\delta W = 0 \qquad (25)$$

The result given by Equation (25) is referred to as the **principle of virtual work.**

Let us consider the presence of constraints in the system. We then have the force on any particle as a vector sum of the applied force $\vec{F}^{(a)}$ and the force of constraint $\vec{F}^{(c)}$. Thus, we get

$$\vec{F}_k = \vec{F}_k^{(a)} + \vec{F}_k^{(c)} \qquad (26)$$

Equation (24) then becomes

$$\sum_{k=1}^{N} \left(\vec{F}_k^{(a)} + \vec{F}_k^{(c)} \right) \cdot \delta\vec{r}_k = 0$$

or $$\sum_{k=1}^{N} \vec{F}_k^{(a)} \cdot \delta\vec{r}_k + \sum \vec{F}_k^{(c)} \cdot \delta\vec{r}_k = 0 \qquad (27)$$

Unlike in Equation (24), the left hand side of Equation (27) can not be interpreted as the net work done in the virtual displacements of the particles of the system. This is because forces of constraints $\vec{F}_k^{(c)}$'s are not continuous functions of positions.

Let us restrict our considerations to only such systems for which

$$\vec{F}_k^{(c)} \cdot \delta\vec{r}_k \geq 0 \qquad (28)$$

NOTES

For all δr_k which are consistent with the constraints, from Equations (27) and (28), we get

$$\sum_k \vec{F}_k^{(a)} \cdot \delta\vec{r}_k \leq 0 \qquad (29)$$

The only forces involved in Equation (29) are the applied forces which may be considered as continuous functions of positions, in general. We are then in a position to interpret the left hand side of Equation (29) as the net work done by the applied forces during the virtual displacements of the particles consistent with the constraints and express Equation (29) as

$$\delta W = \sum \vec{F}_k^{(a)} \cdot \delta\vec{r}_k \leq 0 \qquad (30)$$

Let the virtual displacements under consideration be restricted to displacements which are reversible in the geometrical sense. Denoting reversible displacements by $\delta'\vec{r}_k$ we get from Equation (29)

$$\sum \vec{F}_k^{(a)} \cdot \delta'\vec{r}_k \leq 0$$

Also,

$$\sum \vec{F}_k^{(a)} \cdot \left(-\delta'\vec{r}_k\right) \leq 0$$

The above two results hold only if

$$\sum \vec{F}_k^{(a)} \cdot \delta'\vec{r}_k = 0 \qquad (31)$$

Equation (31) is the generalized form of the principle of virtual work. The principle can be stated as follows:

The work done in infinitesimal reversible virtual displacements, consistent with the constraints, from the equilibrium configuration of a system is zero.

It is important to note that the equilibrium of the system we have referred to in the above discussion is static equilibrium. If we extend this argument to systems in motion, we obtain another important principle called the D'Alembert's principle.

Check Your Progress

5. What are virtual displacements?
6. Write the equation that we refer for virtual work.
7. Define the principle of virtual work.

2.4 D'ALEMBERT'S PRINCIPLE

NOTES

If $\vec{r}_k$ is the radius vector of the k^{th} particle in the system of particles considered above we have the equation of motion of the particle

$$\vec{F}_k = \frac{d}{dt}\left(m_k \dot{\vec{r}}_k \right) \tag{32}$$

or

$$\vec{F}_k - \frac{d}{dt}\left(m_k \dot{\vec{r}}_k \right) = 0 \tag{33}$$

If $\delta \vec{r}_k$ is an infinitesimal virtual displacement of the particle, we obtain from Equation (33)

$$\left[\vec{F}_k - \frac{d}{dt}\left(m_k \dot{\vec{r}}_k \right) \right] \cdot \delta \vec{r}_k = 0$$

Considering all the particles in the system, the above equation gives

$$\sum_k \left[\vec{F}_k - \frac{d}{dt}\left(m_k \dot{\vec{r}}_k \right) \right] \cdot \delta \vec{r}_k = 0 \tag{34}$$

In the presence of constraints in the system, the above equation can be written as

$$\sum_k \left[\left(\vec{F}_k^{(a)} + \vec{F}_k^{(c)} \right) - \frac{d}{dt}\left(m_k \vec{r}_k \right) \right] \cdot \delta \vec{r}_k = 0$$

or,

$$\sum_k \vec{F}_k^{(c)} \cdot \delta \vec{r}_k + \sum_k \left[\vec{F}_k^{(a)} - \frac{d}{dt}\left(m_k \dot{\vec{r}}_k \right) \right] \cdot \delta \vec{r}_k = 0 \tag{35}$$

Once again restricting our consideration to only such systems for which

$$\vec{F}_k^{(c)} \cdot \delta \vec{r}_k \geq 0$$

for all $\delta \vec{r}_k$ which are compatible with the constraints, we get from Equation (35)

$$\sum_k \left[\vec{F}_k^{(a)} - \frac{d}{dt}\left(\vec{p}_k \right) \right] \cdot \delta \vec{r}_k \leq 0 \tag{36}$$

where $\vec{p}_k = m_k \dfrac{d}{dt}\left(\vec{r}_k \right)$ is the momentum of the k^{th} particle.

NOTES

Further, considering only such virtual displacements which are reversible and denoted as $\delta' \vec{r}_k$ we obtain

$$\sum_k \left[\vec{F}_k^{(a)} - \frac{d}{dt}\left(\vec{p}_k \right) \right] \cdot \delta' \vec{r}_k \leq 0 \tag{37}$$

and also

$$\sum_k \left[\vec{F}_k^{(a)} - \frac{d}{dt}\left(\vec{p}_k \right) \right] \cdot \left(-\delta \vec{r}_k \right) \leq 0 \tag{38}$$

Simultaneous validity of Equations (37) and (38) gives

$$\sum_k \left[\vec{F}_k^{(a)} - \frac{d}{dt}\left(\vec{p}_k \right) \right] \cdot \delta' \vec{r}_k = 0 \tag{39}$$

The term $\sum_k F_k^{(a)} \cdot \delta' \vec{r}_k$ is the net work done by the applied forces in the course of virtual displacements of the particles of the system. It is usual to call $-\frac{d}{dt}\left(\vec{p}_k \right)$ as the **force of inertia** which may be denoted as $\vec{F}_k^{(I)}$. In view of this we may write Equation (39) as

$$\sum_k \left[\vec{F}_k^{(a)} + \vec{F}_k^{(I)} \right] \cdot \delta' \vec{r}_k = 0 \tag{40}$$

$\left(\vec{F}_k^{(a)} + \vec{F}_k^{(I)} \right)$ can be called the *effective force* on the k^{th} particle and denoted as $\vec{F}_k^{\text{eff}}$. We can then write Equation (65) as

$$\sum_k \vec{F}_k^{\text{eff}} \cdot \delta' \vec{r}_k = 0 \tag{41}$$

Equations (39), (40) and (41) are different mathematical forms of D'Alembert's principle. The principle may be stated as follows:

For any dynamical system, the total work done by the effective force is zero in the course of reversible infinitesimal virtual displacement compatible with the constraints imposed on the system.

It is important to note that the coefficients of $\delta' \vec{r}_k$ in Equation (39), (40) and (41) cannot be put equal to zero because $\delta' \vec{r}_k$ are not independent of each other.

2.5 LAGRANGE'S EQUATIONS FOR GENERALIZED/CONSERVATIVE SYSTEMS

NOTES

Consider a mechanical system of N particles. At some instant of time t, let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ be the position vectors of the particles with respect to some fixed origin. If the system is described by s generalized coordinates $q_1, \ldots, q_s$, then we have the transformation equations

$$\vec{r}_i = \vec{r}_i(q_1, \ldots, q_s, t)\;.\;(i = 1, \ldots, N) \qquad (42)$$

Velocity vectors for the particles are then given by

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \sum_{k=1}^{s} \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \qquad (43)$$

If $\delta \vec{r}_i$ is an infinitesimal virtual displacement of the i^{th} particle, we get

$$\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j + \frac{\partial \vec{r}_i}{\partial t}\delta t$$

However, since virtual displacement does not refer to displacement with respect to time, we have $\dfrac{\partial \vec{r}_i}{\partial t} = 0$ and hence we obtain

$$\delta \vec{r}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j \qquad (44)$$

According to D'Alembert's principle we have

$$\sum_i \left(\vec{F}_i - \dot{\vec{p}}_i\right).\delta \vec{r}_i = 0 \qquad (45)$$

where $\vec{F}_i$ is the actual force acting on the i^{th} particle and $\dot{\vec{p}}_i$ is the reverse effective force. Using Equation (44) in Equation (45) we get

$$\sum_i \left(\vec{F}_i - \dot{\vec{p}}_i\right) \cdot \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = 0$$

or
$$\sum_{i,j} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j - \sum_{i,j} \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = 0$$

or
$$\sum_j Q_j \delta q_j - \sum_{i,j} \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = 0 \qquad (46)$$

where $$Q_j = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \tag{47}$$

are the components of the generalized forces.

NOTES

We further have

$$\sum_{i,j} \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

$$= \sum_{i,j} \left[\frac{d}{dt} \left(m_i \vec{r}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) - m_i \vec{r}_i \cdot \frac{d}{dt} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \right] \delta q_j \tag{48}$$

Now

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = \sum_k \frac{\partial}{\partial q_k} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right) \dot{q}_k + \frac{\partial}{\partial t} \left(\frac{\partial \vec{r}_i}{\partial q_j} \right)$$

$$= \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_k \partial q_j} \dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial t \partial q_j}$$

$$= \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t}$$

$$= \frac{\partial}{\partial q_j} \left[\sum \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right]$$

or $$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j} \right) = \frac{\partial}{\partial q_j} \frac{d \vec{r}_i}{dt} = \frac{\partial \vec{v}_i}{\partial q_j} \tag{49}$$

We have from Equation (43)

$$\frac{\partial \vec{v}_i}{\partial q_j} = \frac{\partial}{\partial q_j} \left[\sum_k \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \right]$$

or $$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} \tag{50}$$

Substituting Equations (49) and (50) in Equation (48) we obtain

$$\sum_{i,j} \vec{p}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_{i,j} \left[\frac{d}{dt} \left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} \right] \delta q_j$$

NOTES

$$= \sum_j \left[\frac{d}{dt} \left\{ \frac{\partial}{\partial \dot{q}_j} \left(\sum_i \frac{1}{2} m_i v_i^2 \right) \right\} - \frac{\partial}{\partial q_j} \left(\sum \frac{1}{2} m_i v_i^2 \right) \right] \delta q_j \quad (51)$$

$$= \sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j \quad (52)$$

where $T = \sum \frac{1}{2} m_i v_i^2$ = Kinetic energy of the system of particles.

Using Equation (51) in Equation (46) we then obtain

$$\sum_j Q_j \delta q_j - \sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j = 0$$

The above can be re-written as

$$\sum_j \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0 \quad (53)$$

For holonomic constraints, q_j's are independent of each other and hence the coefficient of each δq_j in Equation (53) separately vanishes giving

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j; \; (j = 1, \ldots.., S) \quad (54)$$

Considering the system under consideration to be conservative, the potential energy U of the system is a function of only the position vectors, i.e., $U = U(\vec{r}_1, \ldots.., \vec{r}_N)$ and force on each particle can be derived from the potential energy function U according to

$$\vec{F}_i = -\frac{\partial U}{\partial \vec{r}_i} \quad (55)$$

The generalized force Q_j, given by Equation (47) can thus be written as

$$Q_j = \sum_{i=1}^{N} -\frac{\partial U}{\partial r_i} \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial \vec{U}}{\partial q_j} \quad (56)$$

In view of Equation (56), we get from Equation (54)

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = -\frac{\partial U}{\partial q_j}$$

or

$$\frac{d}{dt} \frac{\partial (T)}{\partial \dot{q}_j} - \frac{\partial (T - U)}{\partial q_j} = 0$$

Since U does not depend on the generalized velocities, the above equation can also be written as

$$\frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} (T - U) - \frac{\partial}{\partial q_j} (T - U) = 0 \quad (57)$$

If we replace $T - U$ by L, Equation (57) becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_j}\right) - \frac{\partial L}{\partial \dot{q}_j} = 0 \qquad (58)$$

NOTES

Since T is a function of the generalized velocities $\dot{q}_k$'s and U is a function of generalized coordinate q_k's, we find L to be a function of coordinates, velocities and time in general, and in view of Equation (58) we can identify L as the Lagrangian function of the system. Thus, for a conservative system we obtain

L = Kinetic energy of the system – Potential energy of the system

2.5.1 Applications of Lagrangian Formulation

1: Motion of a Simple Pendulum Placed in a Uniform Gravitational Field

A simple pendulum consists of a point mass m at one end of a weightless, inelastic string of length l, the other end being rigidly clamped. The mass m can swing back and forth in a vertical plane about the position of rest once it is displaced from the position of rest and released. As the mass is constrained to move on a circular arc in the vertical plane, the pendulum has only one degree of freedom. Thus, the pendulum is described, at any time, by only one generalized coordinate which can be conveniently taken as θ, the angle the string makes with the vertical as shown in Figure 2.1.

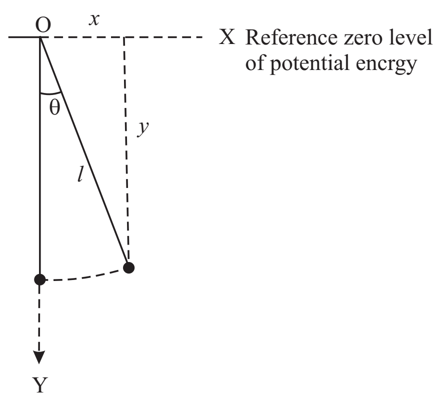

Fig. 2.1 *Movement of Pendulum*

If x and y are the coordinates of the mass point with respect to the origin at the point of suspension O, then we have

$$x = l\sin\theta$$

$$y = l\cos\theta$$

The kinetic energy of the point mass is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left[\left(l\cos\dot{\theta}\right)^2 + \left(-l\sin\dot{\theta}\right)^2\right]$$

or $$T = \frac{1}{2}ml^2\left[\cos^2\theta\dot{\theta}^2 + \sin^2\theta\dot{\theta}^2\right]$$

NOTES

or $$T = \frac{1}{2}ml^2\dot{\theta}^2 \quad (59)$$

The potential energy of the point mass in the gravitational field is

$$V = -mgy = -mgl\cos\theta \quad (60)$$

The Lagrangian of the pendulum is thus given by

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \quad (61)$$

The Lagrange's equation for the coordinate θ is

$$\frac{d}{dl}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

Using L given by Equation (61), the above gives

$$\frac{d}{dt}\left[\frac{1}{2}ml^2 2\dot{\theta}\right] = -mgl\sin\theta$$

or $$\frac{d}{dt}ml\dot{\theta} = mgl\sin\theta$$

or $$l\ddot{\theta} = -g\sin\theta$$

or $$\ddot{\theta} = -\frac{g}{l}\sin\theta \quad (62)$$

Considering θ to be small, we get $\sin\theta \approx \theta$ and hence the above equation reduces to

$$\ddot{\theta} = -\frac{g}{l}\theta \quad (63)$$

Equation (63) shows that under the condition that the angular amplitude is very small the motion of the pendulum is simple harmonic of time period

$$T = 2\pi\sqrt{\frac{l}{g}}$$

2: Motion of a Compound Pendulum in a Uniform Gravitational Field

Any rigid body capable of oscillating in a vertical plane about a horizontal axis passing through any point (excepting the centre of gravity) of the body is called a compound pendulum.

Let the vertical plane of oscillation of the compound pendulum be the XY plane.

Let us choose the origin of the coordinate system as the point O through which the horizontal axis (the X axis) passes.

Let G be the position of the centre of gravity of the body when at rest.

OG = l (say)

On displacing the pendulum slightly from the position of rest and releasing, the pendulum begins to oscillate about the horizontal axis through O.

At any instant of time t, let G' be the new position of the centre of gravity and $G\hat{O}G'$ be equal to θ as shown in Figure 2.2.

The kinetic energy of the pendulum at the instant t is

$$T = \frac{1}{2} I\dot{\theta}^2 \tag{64}$$

where I is the moment of inertia of the pendulum about the axis of oscillation.

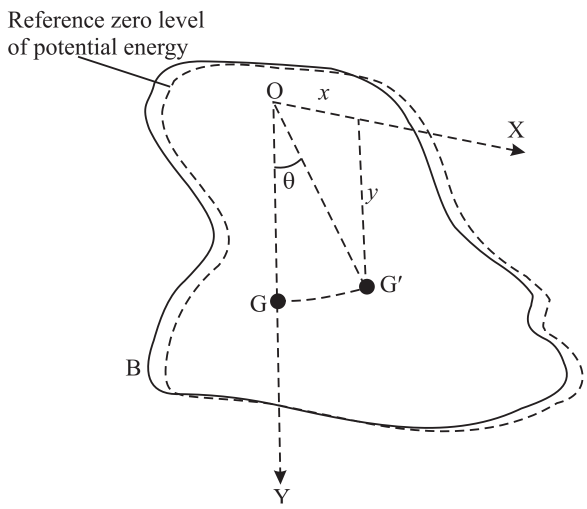

Fig. 2.2 *Positions of Centre of Gravity*

Taking the horizontal axis OX as the reference zero of potential energy, we get the potential energy of the pendulum at the instant t as

$$V = -mgy = -mgl\cos\theta \tag{65}$$

The Lagrangian of the pendulum is thus

$$L = T - V = \frac{1}{2} I\dot{\theta}^2 + mgl\cos\theta \tag{66}$$

From Equation (66) we find that the only generalized coordinate for the pendulum is θ. We thus have the Lagrange's equation for the compound pendulum

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \tag{67}$$

Using L given by Equation (66) the above equation gives

$$\frac{d}{dt}\left(\frac{1}{2} I 2\dot{\theta}\right) + mgl\sin\theta = 0$$

or $$I\ddot{\theta} + mgl\sin\theta = 0$$

or $$\ddot{\theta} + \frac{mgl}{I}\sin\theta = 0 \tag{68}$$

Considering θ small Equation (68) reduces to

$$\ddot{\theta} = -\frac{mgl}{I}\theta \qquad (69)$$

NOTES

Clearly, the motion of the pendulum is simple harmonic of time period

$$T = 2\pi\sqrt{\frac{I}{mgl}} \qquad (70)$$

3: Motion of a Spherical Pendulum

A spherical pendulum consists of a point mass *m* constrained to move on the surface of a sphere. The position of the point mass at any instant is located by the Cartesian coordinates *x, y, z*, or more conveniently by the spherical polar coordinates *r*, θ, ϕ (*r* = radius of the sphere is constant) with respect to a coordinate frame XYZ having the origin at the centre of the sphere as shown in Figure 2.3.

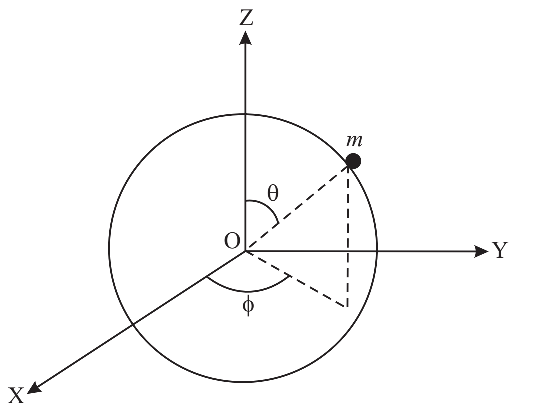

Fig. 2.3 *Motion of Spherical Pendulum*

We have the transformation equations

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \qquad (71)$$

The kinetic energy of the body at the instant of time under consideration is given by

$$T = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right] \qquad (72)$$

$\dot{x}, \dot{y}$ and $\dot{z}$ found by differentiating Equation (*i*) with respect to time, when substituted in Equation (72) gives

$$T = \frac{1}{2}m\dot{r}^2\left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right] \qquad (73)$$

NOTES

Considering the horizontal plane XOY as the plane of zero potential energy, we get the potential energy of the body as

$$V = mgz = mgr\cos\theta \tag{74}$$

The Lagrangian of the spherical pendulum is then given by

$$L = T - V = \frac{1}{2}m\dot{r}^2\left[\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right] - mgr\cos\theta \tag{75}$$

From Equation (75) we find the generalized coordinates for the pendulum to be θ and ϕ (since r is constant), so that the Lagrange's equations are

(*i*) $$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = \frac{\partial L}{\partial\theta}$$

or $$\frac{d}{dt}\left[\frac{1}{2}mr^2 2\dot{\theta}\right] = \frac{1}{2}mr^2\dot{\phi}^2 2\sin\cos\theta + mgr\sin\theta$$

or $$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - mr^2\sin\cos\theta\dot{\phi}^2 - mgr\sin\theta = 0$$

or $$r\ddot{\theta} - r\sin\cos\theta\dot{\phi}^2 - g\sin\theta = 0 \tag{76}$$

(*ii*) $$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi}$$

or $$\frac{d}{dt}\left[\frac{1}{2}mr^2\sin^2\theta 2\dot{\phi}\right] = 0$$

or $$mr^2\frac{d}{dt}\left[\sin^2\theta\dot{\phi}\right] = 0 \tag{77}$$

4: Motion of a Particle Under a Central Force

Central force is that force which acts either towards or away from a fixed point (called the centre of the force) and depends only on the distance r from the fixed point.

We may thus express the magnitude of central force as $F = F(r)$.

Further, any central force can be derived from a potential function V according to

$$F = -\frac{dV}{dr}$$

which gives $dV = -Fdr$ and hence

$$V = -\int Fdr$$

Since F depends only on the distance r, we find from the above, the potential V to depend only on r, i.e., on the distance from the force centre. Thus, $V = V(r)$.

The most important characteristic of motion of a particle under central force is that the motion is restricted to take place in a plane. The number of degrees of freedom of the particle is thus two and the convenient generalized coordinates are polar coordinates r and θ as indicated in Figure 2.4.

NOTES

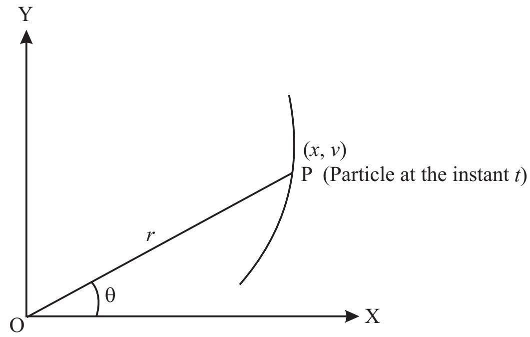

Fig. 2.4 *Polar Coordinates r and* α

The kinetic energy of the particle at the instant t is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \tag{78}$$

Coordinate transformation equations are

$$x = r\cos\theta$$

and

$$y = r\sin\theta \tag{79}$$

Equations (79) give

$$\dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta} \tag{80}$$

$$\dot{y} = \dot{r}\sin\theta + r\cos\theta\dot{\theta} \tag{81}$$

Using Equations (80) and (81) in Equation (78) we obtain

$$T = \frac{1}{2}m\left[r^2\cos^2\theta + r^2\sin^2\theta\dot{\theta}^2 - 2r\dot{r}\cos\theta\sin\theta\dot{\theta} + r^2\sin^2\theta + r^2\cos^2\theta\dot{\theta}^2 + 2r\dot{r}\sin\theta\cos\theta\dot{\theta}\right]$$

or

$$T = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2\right] \tag{82}$$

The potential energy of the particle is

$$V = V(r) \tag{83}$$

Thus, the Lagrangian of the particle is given by

$$L = T - V = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2\right] - V(r) \tag{84}$$

Lagrange's equations are

(a) $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{r}}\right) = \dfrac{\partial L}{\partial r}$

Using L given by Equation (84), the above gives

$$\frac{d}{dt}\left[\frac{1}{2}m2\dot{r}\right] = \frac{1}{2}m\dot{\theta}^2 2r - \frac{dV(r)}{dr}$$

NOTES

$$\text{or} \qquad m\ddot{r} = mr\dot{\theta}^2 - \frac{dV(r)}{dr}$$

$$\text{or} \qquad m\ddot{r} - mr\dot{\theta}^2 = -\frac{dV(r)}{dr} \qquad (85)$$

$$(b) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

Using L given by Equation (84), the above gives

$$\frac{d}{dt}\left[\frac{1}{2}mr^2 2\dot{\theta}\right] = 0 \qquad (86)$$

$$\frac{d}{dt}\left(\frac{1}{2}mr^2\dot{\theta}\right) = 0$$

$$\text{or} \qquad mr^2\dot{\theta} = \text{constant} \qquad (87)$$

Note: By definition, the generalized momentum (p_q) conjugate to the generalized coordinate θ is given by

$$p_q = \frac{\partial L}{\partial \dot{\theta}}$$

Using Equation (84), the above becomes

$$p_q = mr^2\dot{\theta} \qquad (88)$$

Equations (87) and (88) show that for a particle moving under a central force, the angular momentum is a constant quantity.

From Equation (86) we get

$$m2r\dot{r}\dot{\theta} + mr^2\ddot{\theta} = 0$$

$$\text{or} \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \qquad (89)$$

5: Motion of a Linear Harmonic Oscillator

Let a particle mass m undergo simple harmonic motion along the X-axis. Let us measure displacement of the particle from the mean position O which is taken as the origin of the X-axis as shown in Figure 2.5.

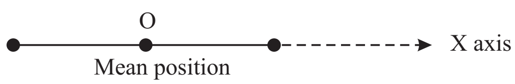

Fig. 2.5

If x is the displacement of the particle at any instant of time t, the kinetic energy of the particle at that instant is

$$T = \frac{1}{2}m\dot{x}^2 \qquad (90)$$

If k is the restoring force per unit displacement acting on the particle then the potential energy of the particle when the displacement is x is given by

$$V = \frac{1}{2}kx^2 \qquad (91)$$

NOTES

The Lagrangian of the oscillator is thus

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \qquad (92)$$

The Lagrange's equation for the oscillator is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \qquad (93)$$

Using Equation (92) we get

$$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m2\dot{x} = m\dot{x} \qquad (94)$$

and

$$\frac{\partial L}{\partial \dot{x}} = -\frac{1}{2}k2x = -kx \qquad (95)$$

Using Equations (94) and (95) in Equation (93) we obtain

$$\frac{d}{dt}(m\dot{x}) = -kx$$

or

$$m\ddot{x} + kx = 0$$

which is the familiar equation of motion for a linear harmonic oscillator.

6: Motion of a Coplanar Double Pendulum Placed in a Uniform Gravitational Field

A double pendulum consists of two mass points m and m_1 at the ends of two weightless rods of lengths l and l_1.

The rod of length l is suspended from a rigid support at O while the rod of length l_1 is suspended from a hinge at the mass point m as shown in Figure 2.6.

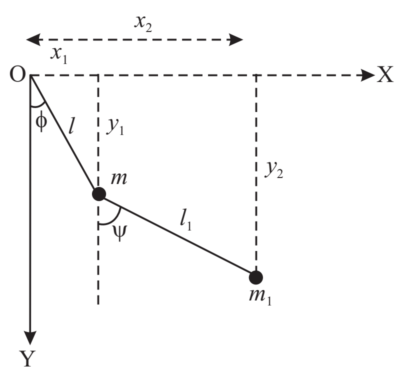

Fig. 2.6 *Pendulum Motion: In Uniform Gravitational Field*

Both the pendulums are constrained to move in the same vertical plane, say the XY plane and hence the number of degrees of freedom for the pendulum is two.

The two generalized coordinates are conveniently chosen as angles ϕ and ψ which the two rods make with the Y-axis which is assumed to be vertical.

NOTES

The Cartesian coordinates of the mass point m are

$$x_1 = l \sin \phi;\ y_1 = l \cos \phi \tag{96}$$

The Cartesian coordinates of the mass point m_1 are

$$x_2 = l \sin \phi + l_1 \sin \psi;\ y_2 = l_1 \cos \phi + l_2 \cos \psi \tag{97}$$

For the mass point m, the kinetic energy is

$$T_1 = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right) = \frac{1}{2}m\left[\left(l\cos\phi\dot{\phi}\right)^2 + \left(-l\sin\phi\dot{\phi}\right)^2\right] = \frac{1}{2}ml^2\dot{\phi}^2 \tag{98}$$

The potential energy of the mass point m is

$$V_1 = -mgy_1 = -mgl\cos\phi \tag{99}$$

For the mass point m_1, the kinetic energy is

$$T_2 = \frac{1}{2}m_1\left(\dot{x}_2^2 + \dot{y}_2^2\right) = \frac{1}{2}m_1\left[\left(l\cos\phi\dot{\phi} + l_1\sin\psi\dot{\psi}\right)^2 + \left(-l\sin\phi\dot{\phi} - l_1\sin\psi\dot{\psi}\right)^2\right]$$

$$= \frac{1}{2}m_1\left[l^2\dot{\phi}^2 + l^2\dot{\psi}^2 + 2ll_1\cos(\phi-\psi)\dot{\phi}\dot{\psi}\right] \tag{100}$$

The potential energy of the mass m_1 is

$$V_2 = m_1gy_2 = m_1g(l\cos\phi + l_1\cos\psi) \tag{101}$$

The Lagrangian of the double pendulum is thus

$$L = T_1 - V_1 + T_2 - V_2$$

Using Equations (98), (99), (100) and (101) we obtain

$$L = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi + \frac{1}{2}m_1\left[l^2\dot{\phi}^2 + l_1^2\dot{\psi}^2 + 2ll_1\cos(\phi-\psi)\dot{\phi}\dot{\psi}\right] + m_1g\left(l\cos\phi + l_1\cos\psi\right)$$

or

$$L = \frac{1}{2}(m+m_1)l^2\dot{\phi}^2 + \frac{1}{2}m_1l_1^2\dot{\psi}^2 + m_1ll_1\dot{\phi}\dot{\psi}\cos(\phi-\psi) + (m+m_1)gl\cos\phi + m_1gl_1\cos\psi \tag{102}$$

The Lagrange's equations for the pendulum are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = \frac{\partial L}{\partial\phi};\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\psi}}\right) = \frac{\partial L}{\partial\psi} \tag{103}$$

We obtain using Equation (102)

$$(a)\quad (m+m_1)l^2\ddot{\phi} + m_1ll_1\ddot{\psi}\cos(\phi-\psi) + m_1ll_1\dot{\psi}^2\sin(\phi-\psi) + (m+m_1)gl\sin\phi = 0 \tag{104}$$

and

$$(b)\quad m_1l^2\ddot{\phi} + m_1ll_1\ddot{\phi}\cos(\phi-\psi) - m_1ll_1\dot{\phi}^2\sin(\phi-\psi) + m_1gl_1\sin\psi = 0 \tag{105}$$

In the special case when $m_1 = m = m_0$, and $l_1 = l = l_0$, the above equations assume the simple forms

$$2\ddot{\phi} + \ddot{\psi}\cos(\phi-\psi) + \dot{\psi}^2\sin(\phi-\psi) + 2\frac{g}{l_o}\sin\phi = 0 \tag{106}$$

$$\ddot{\psi}+\ddot{\phi}\cos(\phi-\psi)-\dot{\phi}^2\sin(\phi-\psi)+\frac{g}{l_o}\sin\psi = 0 \qquad (107)$$

NOTES

Further, if we consider both ϕ and ψ small, the above equations further reduce to

$$2\ddot{\phi}+\ddot{\psi}+2\frac{g}{l_o}\phi = 0 \qquad (108)$$

$$\ddot{\psi}+\ddot{\phi}+\frac{g}{l_o}\psi = 0 \qquad (109)$$

The above are coupled differential equations for the double pendulum.

7: Lagrangian of a Hoop Rolling Down an Inclined Plane without Slipping

Consider a hoop (circular ring) of radius r and mass m rolling down an inclined plane without slipping (velocity of the instantaneous point of contact of the hoop along the plane is zero) as shown in Figure 2.7.

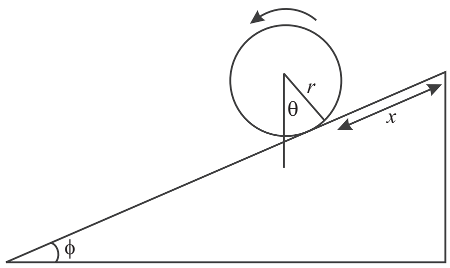

Fig. 2.7 *Motion in an Inclined Plane*

Let us measure the displacement of the centre of mass of the hoop from the top of the incline.

Let at some instant of time t, the centre of mass be at a distance x from the top.

The velocity of the centre of mass is then

$$v = \dot{x} \qquad (110)$$

Since there is no slipping, the angular velocity of the hoop about the axis of rotation through the centre of mass is

$$\dot{\theta} = \frac{v}{r}=\frac{\dot{x}}{r} \qquad (111)$$

Now, the kinetic energy of the hoop at the instant t is

$$T = \frac{1}{2}m\dot{x}^2+\frac{1}{2}mr^2\dot{\theta}^2 \qquad (112)$$

The potential energy of the hoop at the instant t is

$$V = mg(l-x)\sin\phi \qquad (113)$$

where l is the length of the inclined plane.

The Lagrangian of the hoop is thus given by

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mg(l-x)\sin\phi \quad (114)$$

Using Equation (111) we may write the Lagrangian as

NOTES

$$L = m\dot{x}^2 + mgx\sin\phi - mgl\sin\phi \quad (115)$$

8: Lagrangian of a Charged Particle Moving in an Electromagnetic Field

The electric field vector $\vec{E}$ and the magnetic field vector $\vec{B}$ which describe an electromagnetic field satisfy the Maxwell's equations

$$\text{curl}\,\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0\ ;\ \text{div}\,\vec{D} = 4\pi\vec{P}$$

$$\text{curl}\,\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{j}\ ;\ \text{div}\,\vec{B} = 0 \quad (116)$$

The force $\vec{F}$ experienced by a particle of mass m having charge q moving with a velocity $\vec{v}$ in the electromagnetic field is given by

$$\vec{F} = q\vec{E} + \frac{q}{c}\left(\vec{v}\times\vec{B}\right) \quad (117)$$

The electromagnetic field can alternatively be described by a scalar potential ϕ and vector potential $\vec{A}$ defined according to

$$\vec{B} = \text{curl}\ \vec{A}$$

and

$$\vec{E} = -\vec{\nabla}\phi = \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \quad (118)$$

In terms of ϕ and $\vec{A}$ the force $\vec{F}$ becomes

$$\vec{F} = q\left[-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} + \frac{1}{c}\left(\vec{v}\times\vec{\nabla}\times\vec{A}\right)\right] \quad (119)$$

The x-component of $\vec{F}$ is

$$F_x = q\left[-\left(\vec{\nabla}\phi\right)_x - \frac{1}{c}\left(\frac{\partial\vec{A}}{\partial t}\right)_x + \frac{1}{c}\left(\vec{v}\times\vec{\nabla}\times\vec{A}\right)_x\right] \quad (120)$$

Now,

$$(\vec{\nabla}\phi)_x = \frac{\partial\phi}{\partial x} \quad (121)$$

$$\frac{d}{dt}A_x = \frac{\partial}{\partial t}A_x + \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_y}{\partial y}\frac{dy}{dt} + + \frac{\partial A_z}{\partial z}\frac{dz}{dt} \quad (122)$$

$$= \frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_y}{\partial y}v_y + \frac{\partial A_z}{\partial z}v_z$$

NOTES

Also, $$\left(\vec{v}\times\vec{\nabla}\times\vec{A}\right)_x = v_y\left(\vec{\nabla}\times\vec{A}\right)_z - v_z\left(\vec{\nabla}\times\vec{A}\right)_y$$

$$= v_y\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)-v_z\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)$$

$$= v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_y}{\partial y}+v_z\frac{\partial A_z}{\partial z}-v_y\frac{\partial A_x}{\partial y}-v_z\frac{\partial A_x}{\partial z}-v_x\frac{\partial A_x}{\partial x}$$

$$= \frac{\partial}{\partial x}\left(\vec{v}\,.\,\vec{A}\right)-\frac{dA_x}{dt}+\frac{\partial}{\partial t}A_x \qquad (123)$$

Using Equations (121) and (123) in Equation (120) we obtain

$$F = q\left[-\frac{\partial\phi}{\partial x}-\frac{1}{c}\frac{\partial A_x}{\partial t}+\frac{1}{c}\frac{\partial}{\partial x}\left(\vec{v}\,.\,\vec{A}\right)-\frac{1}{c}\frac{dA_x}{dt}+\frac{1}{c}\frac{\partial A_x}{\partial t}\right]$$

$$= \left[-\frac{\partial}{\partial x}\left(\phi-\frac{1}{c}\vec{v}\,.\vec{A}\right)-\frac{1}{c}\frac{d}{dt}\left\{\frac{\partial}{\partial v_x}\left(\vec{v}\,.\vec{A}\right)\right\}\right]$$

or $$F = -\frac{\partial U}{\partial x}+\frac{d}{dt}\frac{\partial U}{\partial v_x} \qquad (124)$$

where $$U = q\phi-\frac{q}{c}\vec{v}\,.\vec{A} \qquad (125)$$

We note that $\frac{\partial\phi}{\partial v_x}=0$ since ϕ is independent of velocities. Thus, U is a kind of generalized velocity-dependent potential.

The Lagrangian of the charged particle is thus given by

$$L = T - U \text{ } (T \text{ is the kinetic energy})$$

or $$L = T-q\phi+\frac{q}{c}\vec{v}\,.\vec{A} \qquad (126)$$

Check Your Progress

8. Define D' Alembert's principle.
9. What would be the Lagrangian function of the system for a conservative system?

2.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. The following set of s number of second order differential equations satisfied by the Lagrangian of the system are called the Lagrange's equations of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=0; \quad (k = 1, 2, ..., s)$$

NOTES

2. The Lagrangian function of a system having s degrees of freedom and described by the generalized coordinates $q_1, \ldots, q_s$ and the generalized velocities $\dot{q}_1, \ldots, \dot{q}_s$ is given by

$$L = L(q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s, t) = L(q_k, \dot{q}_k, t)$$

3. The arbitrary function $F(q_1, \ldots, q_s, t) = F(q_k, t)$ is called **gauge function**.
4. Law of inertia states that, a particle which moves without the influence of any external agent has a constant velocity vector.
5. The corresponding change in the configuration of the system subjected to arbitrary displacement is independent of time, i.e., no actual displacement of the system occurs with respect of time. Such displacements in the configuration space are called virtual displacements.
6. $\delta W = 0$
7. The principle can be stated as follows:

   The work done in infinitesimal reversible virtual displacements, consistent with the constraints, from the equilibrium configuration of a system is zero.
8. The principle may be stated as follows:

   For any dynamical system, the total work done by the effective force is zero in the course of reversible infinitesimal virtual displacement compatible with the constraints imposed on the system.
9. We can identify L as the Lagrangian function of the system. Thus, for a conservative system we obtain

   L = Kinetic energy of the system – Potential energy of the system

2.7 SUMMARY

- The Lagrangian of a system cannot be defined uniquely, but can be defined only within an additive total time derivative of any function of coordinates relevant to the system and time.
- If the Lagrangian is multiplied by any arbitrary constant then the equations of motion remain unaltered.
- A particle which moves without the influence of any external agent has a constant velocity vector.
- The corresponding change in the configuration of the system is independent of time, i.e., no actual displacement of the system occurs with respect to time.
- The work done in infinitesimal reversible virtual displacements, consistent with the constraints, from the equilibrium configuration of a system is zero.

2.8 KEY WORDS

NOTES

- **Law of inertia:** A particle which moves without the influence of any external agent has a constant velocity vector.
- **Virtual displacement:** A presumed infinitesimal change of system coordinates occurring while time is confined constant is known as virtual displacement.
- **Central force:** Central force is that force which acts either towards or away from a fixed point (called the centre of the force) and depends only on the distance from the fixed point.
- **Harmonic oscillator:** It is a system that, when displaced from its equilibrium position, encounters a restoring force proportional to the displacement.
- **Simple harmonic motion:** It is an oscillatory motion under a retarding force proportional to the quantity of displacement from an equilibrium position.

2.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Give a brief account of important properties of Lagrangian function.
2. Describe Lagrangian of a particle moving freely in space.
3. Write a short note on virtual work.
4. Describe motion of a linear harmonic oscillator.
5. Discuss Lagrangian of a charged particle moving in an electromagnetic field.

Long-Answer Questions

1. Derive Lagrange's equations for simple systems.
2. Deduce different mathematical forms of D' Alembert's principle.
3. Discuss applications of Lagrangian formulations.
4. Describe Lagrange's equations for conservative systems.

2.10 FURTHER READINGS

NOTES

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UNIT 3 HAMILTON EQUATION

NOTES

Structure

3.0 INTRODUCTION

In Hamiltonian mechanics, a classical physical system is described by a set of canonical coordinates where each component of the coordinate is indexed to the frame of reference of the system. Hamilton's equations can be derived by looking at how the total differential of the Lagrangian depends on time, generalized positions and generalized velocities. Ignorable coordinate is a generalized coordinate of a mechanical system that does not appear in the system's Lagrangian function or other characteristics functions. Its presence simplifies the integration of the corresponding differential equations of motion of a mechanical system. Routhian mechanics is a hybrid formulation of Lagrangian mechanics and Hamiltonian mechanics developed by Edward John Routh. The difference between the Lagrangian, Hamiltonian, and Routhian functions are their variables. The Routhian approach has the best of both approaches, because cyclic coordinates can be split off to the Hamiltonian equations and eliminated, leaving behind the non-cyclic coordinates to be solved from the Lagrangian equations. In this unit you will Discuss Hamilton's equations and general conservation theorem. You will describe ignorable coordinate interpreting conjugate momenta. You will explain the Routhian function and applications of Hamiltonian dynamics.

3.1 OBJECTIVES

NOTES

After going through this unit, you will be able to:

- Explain Hamilton's equations and general conservation theorem
- Deduct canonical equations from variational principle
- Understand superiority of Hamiltonian formulation over Lagrangian formulation
- Describe ignorable coordinate interpreting conjugate momenta
- Explain the Routhian function and applications of Hamiltonian dynamics

3.2 HAMILTON'S EQUATIONS

In Hamiltonian mechanics, a classical physical system is described by a set of canonical coordinates $r = (q, p)$, where each component of the coordinate q_i, p_i is indexed to the frame of reference of the system. The time evolution of the system is uniquely defined using the Hamilton's equations.

The Hamiltonian of a system is defined to be the sum of the kinetic and potential energies expressed as a function of positions and their *conjugate momenta*.

Conjugate Momenta

The momentum of a particle p_i is defined in terms of its velocity r_i by,

$$p_i = m_i r_i$$

Basically, the common definition of conjugate momentum that is valid for any set of coordinates is given in terms of the Lagrangian as,

$$P_i = \frac{\partial L}{\partial r_i}$$

These two definitions are equivalent for Cartesian variables. In terms of Cartesian momenta, the kinetic energy is given by,

$$K = \sum_{i=1}^{N} \frac{P_i^2}{2m_i}$$

Then, the Hamiltonian, which is defined to be the sum, K+U, expressed as a function of positions and momenta is given by,

$$H\left(p,r\right) = \sum_{i=1}^{N} \frac{2_i^2}{2m_i} + U\left(r_1,\cdots,r_N\right) = H\left(p,r\right)$$

NOTES

where P ≡ P1, ..., PN . In terms of the Hamiltonian, the equations of motion of a system are given by *Hamilton's equations*:

$$r_i = \frac{\partial H}{\partial P_i} \qquad p_i = -\frac{\partial H}{\partial r_i}$$

For cyclic coordinate, the conjugate momentum $\boldsymbol{p_i}$ is constant. The solution of Hamilton's equations of motion will yield a trajectory in terms of positions and momenta as functions of time.

3.2.1 Deduction of Canonical Equations from Variational Principle

Consider a mechanical system of s degrees of freedom. Let (q^1, ….., q^s) be the generalized coordinates, $(\dot{q}_1, ..., \dot{q}_s)$ be the generalized velocities and (p^1, ….., p^s) be the generalized momenta for the system. The Lagrangian L of the system then is given by

$$L = L(q_1, ..., q_s, \dot{q}_1, ..., \dot{q}_s, t) = L(q_k, \dot{q}_k, t) \qquad (1)$$

Hamilton's variational principle, or the Principle of Least Action, is stated as

$$\delta S = \delta\int_{t_1}^{t_2} L(q_k, \dot{q}_k, t)\, dt = 0 \qquad (2)$$

The Hamiltonian function $H = H(q^1, ….., q^s, p^1, ….., p^s, t) = H(q^k, p^k, t)$ is related to the Lagrangian L as

$$H = \sum_{k=1}^{s} p_k \dot{q}_k - L \qquad (3)$$

Using Equation (3) in Equation (2) we obtain

$$\delta S = \delta\int_{t_1}^{t_2}\left[\sum_k p_k \dot{q}_k - H(q_k, p_k, t)\right] dt = 0 \qquad (4)$$

or $$\delta\sum_k \int_{t_1}^{t_2} p_k\, \dot{q}_k\, dt - \delta\int_{t_1}^{t_2} H dt = 0$$

or $$\delta\sum_k \int_{t_1}^{t_2} p_k dq_k - \delta\int_{t_1}^{t_2} H dt = 0 \qquad (5)$$

Equations (4) and (5) are referred to as the **modified Hamilton's Principle**.

Let us label each path of the system in its configuration space between time limits t^1 and t^2 by a parameter α. We can then write the δ variation for the action S as

$$\delta S = d\alpha . \frac{\partial S}{\partial \alpha} \qquad (6)$$

NOTES

The above gives, for generality,

$$\delta \equiv d\alpha . \frac{\partial}{\partial \alpha} \quad (7)$$

Using Equation (4) in the Equation (6), we obtain

$$\delta S = d\alpha . \frac{\partial}{\partial \alpha}\left[\int_{t_1}^{t_2}\left\{\sum_k p_k \dot{q}_k - H(q_k, p_k, t)\right\}dt\right] = 0 \quad (8)$$

It is possible to introduce the differential operator $\frac{\partial}{\partial \alpha}$ inside the integral because the two time limits t^1 and t^2 are the same for all the paths and hence are independent of α. We thus obtain

$$\delta S = d\alpha \cdot \int_{t_1}^{t_2} \sum_k \left\{\frac{\partial p_k}{\partial \alpha}\dot{q}_k + p_k \frac{\partial \dot{q}_k}{\partial \alpha} - \frac{\partial H}{\partial q_k}\frac{\partial q_k}{\partial \alpha} - \frac{\partial H}{\partial p_k}\frac{\partial p_k}{\partial \alpha} - \frac{\partial H}{\partial t}\frac{\partial t}{\partial \alpha}\right\} dt = 0 \quad (9)$$

We now have $\displaystyle\int_{t_1}^{t_2} p_k \frac{\partial \dot{q}_k}{\partial \alpha} dt = \int_{t_1}^{t_2} p_k \frac{d}{dt}\left(\frac{\partial q_k}{\partial \alpha}\right)dt$

Evaluating the integral on the right hand side of the above by parts we obtain

$$\int_{t_1}^{t_2} p_k \frac{\partial \dot{q}_k}{\partial \alpha} dt = \left\{p_k \frac{\partial q_k}{\partial \alpha}\right\}_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_k \cdot \frac{\partial q_k}{\partial \alpha} dt = -\int_{t_1}^{t_2} \dot{p}_k \cdot \frac{\partial q_k}{\partial \alpha} dt \quad (10)$$

since $\frac{\partial q_k}{\partial \alpha} = 0$ at t^1 and at t^2.

Further, we have $$\frac{\partial t}{\partial \alpha} = 0 \quad (11)$$

since the time of travel along all the paths is the same.

Using the results given by Equation (10) and (11) in Equation (9) we obtain

$$d\alpha . \int_{t_1}^{t_2} \sum_k \left[\frac{\partial p_k}{\partial \alpha}\dot{q}_k - \dot{p}_k \frac{\partial q_k}{\partial \alpha} - \frac{\partial H}{\partial q_k}\frac{\partial q_k}{\partial \alpha} - \frac{\partial H}{\partial p_k}\frac{\partial p_k}{\partial \alpha}\right] dt = 0 \quad (12)$$

In view of Equation (7) we may write

$$d\alpha \cdot \frac{\partial p_k}{\partial \alpha} = \delta p^k$$

$$d\alpha \cdot \frac{\partial q_k}{\partial \alpha} = \delta q^k \quad (13)$$

Using Equation (13) in Equation (12) we get

$$\int_{t_1}^{t_2} \sum_k \left(\delta p_k \dot{q}_k - \dot{p}_k \delta q_k - \frac{\partial H}{\partial q_k}\delta q_k - \frac{\partial H}{\partial p_k}\delta p_k\right) dt = 0$$

or, $$\int_{t_1}^{t_2} \sum_k \left\{\left(\dot{q}_k - \frac{\partial H}{\partial p_k}\right)\delta p_k - \left(\dot{p}_k + \frac{\partial H}{\partial q_k}\right)\delta q_k\right\} dt = 0 \quad (14)$$

Since q^k and p^k are independent variables, their variations δq^k and δp^k are also independent. Hence, for Equation (14) to hold, the coefficients of δp^k and δq^k must separately vanish. We hence obtain

$$\dot{q}_k - \frac{\partial H}{\partial p_k} = 0 \qquad \text{or} \qquad \dot{q}_k = \frac{\partial H}{\partial p_k} \tag{15}$$

$$\dot{p}_k + \frac{\partial H}{\partial q_k} = 0 \qquad \text{or} \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

For each k, we have two equations of the form given by Equation (15). These $2s$ number of first-order differential equations are **Hamilton's equations of motion, also called Hamilton's canonical equations**.

3.2.2 Alternative Derivation of Canonical Equations of Motion

Canonical equations of motion can be derived alternatively, using the definitions of Lagrangian function, Hamiltonian function and Lagrange's equations of motion.

We have the Lagrangian function of the system under consideration given by

$$L = L(q_1, ..., q_s, \dot{q}_1, ..., \dot{q}_s, t)$$

The total differential of L is thus

$$dL = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt \tag{16}$$

The Hamiltonian function of the system is

$$H = H(q^1, \ldots.., q^s, p^1, \ldots.., p^s, t)$$

The total differential of H is thus

$$dH = \sum_k \frac{\partial H}{\partial q_k} dq_k + \sum_k \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt \tag{17}$$

Further, H is related to L according to

$$H = \sum p_k \dot{q}_k - L$$

So, we obtain

$$dH = \sum p_k d\dot{q}_k + \sum \dot{q}_k dp_k - dL$$

Substituting for dL from Equation (16) in the above, we get

$$dH = \sum p_k d\dot{q}_k + \sum \dot{q}_k dp_k - \sum \frac{\partial L}{\partial q_k} dq_k - \sum \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \frac{\partial L}{\partial t} dt \tag{18}$$

Using $p_k = \dfrac{\partial L}{\partial \dot{q}_k}$, the above equation reduces to

$$dH = \sum \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \sum \dot{q}_k dp_k - \sum \frac{\partial L}{\partial q_k} dq_k - \sum \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k - \frac{\partial L}{\partial t} dt$$

or $$dH = \sum \dot{q}_k dp_k - \sum \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial t} dt \qquad (19)$$

The Lagrange's equation is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}$$

or $$\frac{d}{dt}(p_k) = \frac{\partial L}{\partial q_k} \quad \text{or} \quad \dot{p}_k = \frac{\partial L}{\partial q_k} \qquad (20)$$

Equation (20) used in Equation (19) gives

$$dH = \sum \dot{q}_k dp_k - \sum \dot{p}_k dq_k - \frac{\partial L}{\partial t} dt \qquad (21)$$

Comparing the coefficients of dp^k, dq^k and dt on the right hand sides of Equation (17) and (21), we obtain

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad (k = 1, 2 \ldots\ldots, s) \qquad (22)$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}$$

and $$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \qquad (23)$$

Equations (22) are the Hamilton's canonical equations of motion as obtained earlier.

3.2.3 Physical Significance of the Hamiltonian Function: A General Conservation Theorem

Consider a system of N free particles 1, 2, ….., N. Let $\vec{r}_1, \vec{r}_2, \ldots\ldots, \vec{r}_N$ be the position vectors of the particles with respect to the origin of an inertial frame at some instant of time.

If $m^1, m^2, \ldots\ldots, m^N$ be, respectively, the masses of the particles then the kinetic energy of the system is

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\dot{\vec{r}}_1 \cdot \dot{\vec{r}}_i\right) \qquad (24)$$

Considering the generalized coordinates for the system to be $q^1, q^2, \ldots\ldots, q^s$ (here $s = 3N$), we have the transformation equations

$$\vec{r}_i = \vec{r}_i(q_1, \ldots\ldots, q_s, t)\,; (i = 1, 2, \ldots\ldots, N) \qquad (25)$$

So, we get

$$\dot{\vec{r}}_i = \sum_{k=1}^{s} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t} \qquad (26)$$

In view of Equation (26), the kinetic energy given by Equation (24) becomes

NOTES

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\sum_{k=1}^{s} \frac{\partial \vec{r_i}}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r_i}}{\partial t} \right) \cdot \left(\sum_{k=1}^{s} \frac{\partial \vec{r_i}}{\partial q_k} \dot{q}_k + \frac{\partial \vec{r_i}}{\partial t} \right)$$

$$= \sum_{i=1}^{N} \frac{1}{2} m_i \left(\sum_k \sum_j \frac{\partial \vec{r_i}}{\partial q_k} \frac{\partial \vec{r_i}}{\partial q_j} \dot{q}_k \dot{q}_j \right) + \sum m_i \left(\sum_k \frac{\partial \vec{r_i}}{\partial q_k} \cdot \frac{\partial \vec{r_i}}{\partial t} \dot{q}_k \right) + \sum_i \frac{1}{2} m_i \left(\frac{\partial \vec{r_i}}{\partial t} \right)^2 \qquad (27)$$

If the transformation equations do not depend on time explicitly then we get

$$\frac{\partial \vec{r_i}}{\partial t} = 0 \qquad (28)$$

and the expression for T given by Equation (27) reduces to

$$T = \sum \frac{1}{2} m_i \left(\sum_k \sum_j \frac{\partial \vec{r_i}}{\partial q_k} \cdot \frac{\partial \vec{r_i}}{\partial q_j} \dot{q}_k \dot{q}_j \right) \qquad (29)$$

We thus observe that the kinetic energy is a homogeneous quadratic function of the generalized velocities for systems in which constraints involved are scleronomous, i.e., independent of time.

Euler's formula for any arbitrary homogeneous function $f = f(x^1, x^2, ..., x^N)$ of m^{th} degree is

$$\sum_{l=1}^{N} \frac{\partial f}{\partial x_l} x_l = mf \qquad (30)$$

Since kinetic energy T is homogeneous quadratic function (degree 2) of generalized velocities, we get according to Equation (30)

$$\sum_{j=1}^{s} \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j = 2T \qquad (31)$$

Restricting our considerations to conservative systems only, the potential energy function V for the system becomes independent of velocities. Hence, we have

$$\frac{\partial V}{\partial \dot{q}_j} = 0 \qquad (32)$$

By definition and by using Equation (32) we further have

$$p^j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}(T - V) = \frac{\partial T}{\partial \dot{q}_j} \qquad (33)$$

We then get the Hamiltonian function H as

$$H = \sum p_j \dot{q}_j - L = \sum_j \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j - L$$

Using Equation (31), the above becomes

$$H = 2T - L = 2T - (T - V)$$

or
$$H = T + V = E \tag{34}$$

We thus find that for a conservative system in which constraints are independent of time, the Hamiltonian function is equal to the total energy E of the system.

General Conservation Theorem

The total time derivative of the Hamiltonian function $H = H(q, p, t)$ for a system of s degrees of freedom is given by

$$\frac{dH}{dt} = \sum_{k=1}^{s} \frac{\partial H}{\partial q_k}\dot{q}_k + \sum \frac{\partial H}{\partial p_k}\dot{p}_k + \frac{\partial H}{\partial t} \tag{35}$$

Using the canonical Equation (22) and (23) in the above, we get

$$\frac{dH}{dt} = \sum_{k=1}^{s} \frac{\partial H}{\partial q_k}\frac{\partial H}{\partial p_k} - \sum \frac{\partial H}{\partial p_k}\frac{\partial H}{\partial q_k} - \frac{\partial L}{\partial t}$$

or
$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \tag{36}$$

If the Lagrangian of the system does not depend on time explicitly, Equation (36) gives

$$\frac{dH}{dt} = 0$$

or
$$H = \text{A constant of motion} \tag{37}$$

We find that the Hamiltonian function which is equal to the total energy is a constant of motion for a system if it or the Lagrangian of the system does not depend on time explicitly.

3.2.4 Superiority of Hamiltonian Formulation over Lagrangian Formulation

We have the Lagrange's equations for a system of s degrees of freedom given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}; \; k = 1, 2, \ldots.., \text{s} \tag{38}$$

where
$$L = L(q_1, \ldots.., q_s, \dot{q}_1, \ldots.., \dot{q}_s, t)$$

If the coordinate q_k is cyclic, i.e., q_k does not appear in the Lagrangian explicitly, we get

$$\frac{\partial L}{\partial q_k} = 0 \tag{39}$$

and hence Equation (38) gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \quad 0$$

NOTES

or $$\dot{p}_k = 0 \qquad (40)$$

Clearly, the momentum conjugate to the coordinate q^k is a constant or an integral of motion.

From Hamilton's canonical equation, we then get

$$-\frac{\partial H}{\partial q_k} = 0 \text{ or } \frac{\partial H}{\partial q_k} = 0 \qquad (41)$$

We find that the coordinate q^k does not appear in the Hamiltonian function explicitly. Thus, if a coordinate q^k is cyclic or ignorable then the number of independent variables in the Hamiltonian formulation reduces by two, because the Hamiltonian function under this condition becomes

$$H = H\,(q^1, \ldots, q^{k-1}, q^{k+1}, \ldots, q^s, p^1, \ldots, p^{k-1}, \alpha, p^{k+1}, \ldots, p^s, t)$$
(42)

In the above p^k which is constant has been taken to be α. Thus, the system is reduced to one involving $(2s-2)+1 = 2s-1$ independent variables in the Hamiltonian formulation. In the Lagrangian formulation, however, this is not the case because even if the cyclic coordinate q^k does not appear in the Lagrangian function, $\dot{q}_k$, in general, appears so that we will need to deal with all the $2s + 1$ variables.

The Hamiltonian formulation of mechanics is superior to the Lagrangian formulation, mainly because of the identification of Hamiltonian function as representing the total energy of the system. Such simple physical meaning cannot be assigned to the Lagrangian function of a system. For this reason we can gain physical insight into a mechanical problem when it is treated in the Hamiltonian formulation.

In the Lagrangian formulation, the generalized coordinates as well as the generalized velocities are treated on equal footing though we know that generalized velocities are dependent variables because they are total time derivatives of the coordinates. Such equal status to coordinates and velocities are thus not of much sense.

In the Hamiltonian formulation, on the other hand, the generalized coordinates and the conjugate momenta which are independent variables are given equal status. This fact allows the freedom of choosing the coordinates and momenta conveniently and thus allows us to formulate mechanics in a more abstract form.

The equal status given to coordinates and momenta and the identification of the Hamiltonian function as the total energy in many systems have played an important role in the development of quantum mechanics and statistical mechanics.

Check Your Progress

1. Define the Hamiltonian of a system.
2. State the principle of least action.
3. Write Euler's formula for any arbitrary homogeneous function.

NOTES

3.3 IGNORABLE COORDINATES

The cyclic coordinates are the generalized coordinates of a certain physical system that do not occur explicitly in the expression of the characteristic function of these systems. When the corresponding equations of motion are used then for every cyclic coordinate we obtain the integral of motion corresponding to it. For example, if the Lagrange function $L(q_i, \dot{q}_i, t)$, where the q_i are generalized coordinates, the $\dot{q}_i$ generalized velocities, and ***t*** the time, does not contain q_j explicitly, then q_j is a cyclic coordinate, and the ***j***th Lagrange equation has the form,

$$\left(d/dt\right)\left(\partial L/\partial \dot{q}_j\right) = 0$$

Which is for Lagrange equations (in mechanics) and simultaneously gives an integral of motion,

$$\frac{\partial L}{\partial \dot{q}_j} = \text{const.}$$

If a generalized coordinate *qj* does not explicitly occur in the Hamiltonian, then p_j is considered as a constant of motion (meaning, a constant, independent of time for a true dynamical motion). Then, *qj* becomes a linear function of time. Such a coordinate *qj* is called a cyclic coordinate.

3.3.1 Ignorable Coordinate

Ignorable coordinate is a generalized coordinate of a mechanical system that does not appear in the systems of Lagrangian function or other characteristic functions. The presence of ignorable coordinates simplifies the integration of the corresponding differential equations of motion of a mechanical system. For example, if the coordinate q_1 does not appear in the Lagrangian function L then the Lagrange equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial \dot{q}_i} = 0 \qquad i = 1, 2, ..., n$$

NOTES

Takes the form,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=0$$

And instantaneously yields the integral,

$$\frac{\partial L}{\partial \dot{q}_i}=\text{const.}$$

Characteristically, according to the Lagrangian formula if L is independent of q_i, i.e., q_i is a cyclic or ignorable coordinate then in such a case the corresponding conjugated momentum $p_i=\frac{\partial L}{\partial \dot{q}_i}$ is conserved. Similarly, from Hamiltonians equations $\dot{p}_i=\frac{-\partial H}{\partial q_i}$ it follows that if H is independent of q_i then the corresponding p_i is conserved.

This is essentially the same result since,

$$H(\overline{q},\overline{p},t)=\sum_{i=1}^{n} p_i\dot{q}_i(\overline{q},\overline{p},t)-L\langle\overline{q},\dot{\overline{q}},(\overline{q},\overline{p},t),t\rangle$$

This specifies that H is independent of *qi* iff L is independent of *qi*. For cyclic coordinates the Hamiltonian formalism has an advantage over the Lagrangian. It is revealed, for a classical dynamical system with a Lagrangian, that the existence of an ignorable coordinate is equivalent to the vanishing of a certain Lagrangian derivative. Coordinates not appearing in the Hamiltonian H are termed as cyclic or ignorable coordinates.

Check Your Progress

4. What are the cyclic coordinates?
5. What do you understand by ignorable coordinates?

3.4 THE ROUTHIAN FUNCTION

Consider a mechanical system having s degrees of freedom and described by generalized coordinates q^1, ….., q^s, generalized velocities $\dot{q}_1, \ldots, \dot{q}_s$ and the generalized momenta p^1, ….., p^s conjugate to the coordinates.

If a coordinate q^k is cyclic, then q^k does not appear in the Lagrangian function L for the system. The corresponding momentum $p^k=\frac{\partial L}{\partial \dot{q}_k}$ then is a constant of motion, i.e., $\dot{q}_k = 0$. The Hamilton's equation $\dot{p}_k = -\frac{\partial H}{\partial q_k}$ then gives $\frac{\partial H}{\partial q_k} = 0$, so that q^k does not appear explicitly in the Hamiltonian function also.

If n coordinates in the system are cyclic then in the Hamiltonian formulation, the system reduces to one with $(s - n)$ degrees of freedom. However, in the Lagrangian formulation, the degrees of freedom still remain s, because the Lagrangian will contain all the s generalized velocities. We thus find that Hamiltonian formulation of mechanics is advantageous when cyclic coordinates are involved in the system.

Routh made use of such advantage in order to gain mathematical simplification for solving mechanical problems.

The method due to Routh consists in changing the basis from $(q, \dot{q})$ to the basis (q, p) for only those coordinates which are cyclic and obtaining their equations of motion in the Hamiltonian formulation, while the rest of the coordinates are considered in the Lagrangian formulation so that they obey Lagrange's equation of motion.

To illustrate the procedure, let us consider a simple system having only two generalized coordinates q and η. The generalized velocities are $\dot{q}$ and $\dot{\eta}$. Let p be the momentum conjugate to q. Then, the Lagrangian of the system is

$$L = L(q, \eta, \dot{q}, \dot{\eta}) \tag{43}$$

so that the total differential of L is given by

$$dL = \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial \eta} d\eta + \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial \dot{\eta}} d\dot{\eta} \tag{44}$$

Using the Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \quad \text{or } \dot{p} = \frac{\partial L}{\partial q} \tag{45}$$

From Equation (44), we obtain

$$dL = \dot{p}dq + \frac{\partial L}{\partial \eta} d\eta + pd\dot{q} + \frac{\partial L}{\partial \eta} d\dot{\eta}$$

or

$$dL = \dot{p}dq + d(p\dot{q}) - \dot{q}dp + \frac{\partial L}{\partial \eta} d\eta + \frac{\partial L}{\partial \dot{\eta}} d\dot{\eta}$$

or

$$d(L - p\dot{q}) = \dot{p}dq - \dot{q}dp + \frac{\partial L}{\partial \eta} d\eta + \frac{\partial}{\partial \dot{\eta}} d\dot{\eta} \tag{46}$$

The Routhian or Routh function usually denoted by R is defined as

$$R = p\dot{q} - L \tag{47}$$

NOTES

We find from Equation (46)

$$R = R(p, q, \eta, \dot{\eta}) \quad (48)$$

The total differential of R is hence given by

$$dR = \frac{\partial R}{\partial q}dq + \frac{\partial R}{\partial p}dp + \frac{\partial R}{\partial \eta}d\eta + \frac{\partial R}{\partial \dot{\eta}}d\dot{\eta} \quad (49)$$

From Equation (46) we obtain

$$dR = -\dot{p}dq + \dot{q}dp - \frac{\partial L}{\partial \eta}d\eta - \frac{\partial L}{\partial \dot{\eta}}d\dot{\eta}$$

(50)

Comparing Equation (49) and (50) we obtain

$$\dot{p} = -\frac{\partial R}{\partial q} \quad (51)$$

$$\dot{q} = \frac{\partial R}{\partial p} \quad (52)$$

$$\frac{\partial L}{\partial \eta} = -\frac{\partial R}{\partial \eta} \quad (53)$$

and
$$\frac{\partial L}{\partial \dot{\eta}} = -\frac{\partial R}{\partial \dot{\eta}} \quad (54)$$

Now the Lagrange's equation for the coordinate η is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}}\right) = \frac{\partial L}{\partial \eta}$$

Using Equation (53) and (54) in the above, we get

$$\frac{d}{dt}\left(-\frac{\partial R}{\partial \dot{\eta}}\right) = -\frac{\partial R}{\partial \eta}$$

or
$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{\eta}}\right) = \frac{\partial R}{\partial \eta} \quad (55)$$

Equations (51) and (52) tell us that the Routhian R acts as the Hamiltonian with respect to coordinate q, while Equation (55) reveals that R behaves as the Lagrangian with respect to coordinate η.

If now we consider the coordinate q to be cyclic, we have the conjugate momentum p as constant, say α. We then get

$$R = R(\eta, \dot{\eta}, \alpha) \quad (56)$$

Equation of motion given by Equation (51) and (52), then gives

$$\dot{p} = 0$$

and
$$q = \frac{\partial R}{\partial \alpha} \quad (57)$$

Further, we find that Equation (55) does not contain the ignorable coordinates and hence it becomes easier to solve the equation.

NOTES

A problem involving both cyclic as well as non-cyclic coordinates can be solved by solving the Hamilton's equations separately for the cyclic coordinates with the Routhian R as the Hamiltonian of the system, and Lagranges equations for the non-cyclic coordinates separately with the Routhian as the Lagrangian of the system.

3.4.1 Applications of Hamiltonian Dynamics

1. Motion of a Simple Pendulum

Consider a simple pendulum having the bob of mass m and length l as shown in Figure 3.1. In the equilibrium position A of the bob, the string OA is vertical. Let us displace the bob to the position B so that the string takes the position OB.

$$A\hat{O}B = \theta \text{ (say)}$$

The polar coordinates of the bob at the displaced position are i (a constant) and θ.

With respect to the coordinate frame xy as indicated in the figure, let (x, y) be the cartesian coordinates of the bob at the position B. We then have

$$x = l \cos\theta, \quad y = l - l \sin\theta \tag{58}$$

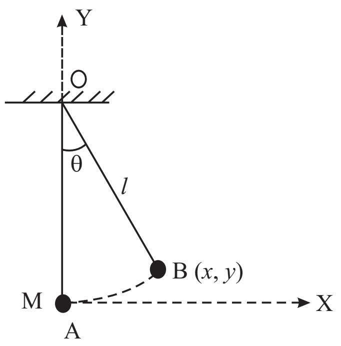

Fig. 3.1 *Simple Pendulum*

The kinetic energy of the bob at the position B is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left[(-l \sin\theta\, \dot{\theta})^2 + (-l \cos\theta\, \dot{\theta})^2\right]$$

or
$$T = \frac{1}{2}ml^2\dot{\theta}^2 \tag{59}$$

With respect to the horizontal drawn from A as the reference zero of potential energy, the potential energy of the bob at the position B is

$$V = mgl\,(1 - \cos\theta) \tag{60}$$

We thus have the Lagrangian of the pendulum as

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\,(1 - \cos\theta) \tag{61}$$

NOTES

The momentum conjugate to the coordinate θ is by definiton

$$p^{q} = \frac{\partial L}{\partial \dot{\theta}} = ml^2\,\dot{\theta} \tag{62}$$

The Hamiltonian for the pendulum is then by definition given by

$$H = p_\theta\,\dot{\theta} - L$$

or
$$H = ml^2\,\dot{\theta}^2 - \frac{1}{2}ml^2\,\dot{\theta}^2 + mgl\,(1-\cos\theta)$$

Substituting for $\dot{\theta}$ from Equation (62) we obtain

$$H = \frac{1}{2}ml^2\,\frac{p_\theta^2}{m^2 l^4} + mgl(1-\cos\theta)$$

or
$$H = \frac{1}{2}\frac{p_\theta^2}{ml^2} + mgl\,(1-\cos\theta) \tag{63}$$

The Hamilton's canonical equations are then

(*a*) $\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$ using Equation (63) (64)

(*b*) $\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = -\,mgl\sin\theta$ using Equation (63) (65)

Using Equation (62) in Equation (65) we obtain

$$ml^2\,\ddot{\theta} = -\,mgl\sin\theta$$

or
$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{66}$$

Assuming a small angular displacement of the bob (θ small) Equation (66) becomes

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{67}$$

The above is the equation of motion for the simple pendulum.

From Equation (67) time period of oscillation of the simple pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}} \tag{68}$$

2. Motion of a Compound Pendulum

Let us consider the compound pendulum as described in Sec. 8.5 (Fig. 8.17). The Lagrangian of the pendulum is given by

$$L = \frac{1}{2}I\dot{\theta}^2 + mgl\cos\theta \tag{69}$$

where I is the moment of inertia of the pendulum about the axis of oscillation.

The momentum conjugate to the generalized coordinate θ is by definition given by

$$p^{q} = \frac{\partial L}{\partial \dot{\theta}} = I\,\dot{\theta} \tag{70}$$

By definition, the Hamiltonian of the pendulum is

$$H = p^{\mathrm{q}}\dot{\theta} - L$$

Using Equation (69) and (70) in the above we get

$$H = \frac{p_\theta^2}{I} - \frac{1}{2} I \frac{p_\theta^2}{I^2} - mgl\cos\theta$$

or
$$H = \frac{1}{2}\frac{p_\theta^2}{I} - mgl\cos\theta \qquad (71)$$

The canonical equations of motion for the pendulum are

$$(a)\quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I} \qquad (72)$$

$$(b)\quad \dot{p}_\theta = \frac{-\partial H}{\partial \theta} = -\,mgl\sin\theta \qquad (73)$$

From Equation (72) we get

$$p^{\mathrm{q}} = I\dot{\theta}$$

and hence
$$\dot{p}_\theta = I\ddot{\theta} \qquad (74)$$

Using Equation (73) in the above we obtain

$$I\ddot{\theta} = -\,mgl\sin\theta$$

$$\ddot{\theta} + \frac{mgl}{I}\sin\theta = 0 \qquad (75)$$

For small angular displacement θ, Equation (75) takes the form

$$\ddot{\theta} + \frac{mgl}{I}\theta = 0 \qquad (76)$$

From Equation (76) we get the time period of oscillation of the compound pendulum to be given by

$$T = 2\pi\sqrt{\frac{I}{mgl}} \qquad (77)$$

3. Motion of a Particle under the Central Force

Motion of a particle under central force has been discussed in the Lagrangian formulation in Section 2.5.1 (Application 4). As has been seen, the Lagrangian of the particle in polar coordinates r and θ is given by

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \qquad (78)$$

The momenta conjugate to the coordinates r and θ are

$$p^{\mathrm{r}} = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad (79)$$

and
$$p^{\mathrm{q}} = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad (80)$$

NOTES

Since the system is conservative, the Hamiltonian is the total energy and is given by

$$H = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \tag{81}$$

Substituting for $\dot{r}$ and $\dot{\theta}$ from Equation (79) and (80) respectively in Equation (81) we obtain

$$H = \frac{1}{2}m\left[\frac{p_r^2}{m^2} + r^2\frac{p_\theta^2}{m^2r^4}\right] + V(r)$$

or
$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r) \tag{82}$$

The cononical equations of motion for the particle are then given by

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{dV(r)}{dr} \tag{83}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \tag{84}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \tag{85}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \tag{86}$$

From Equation (84) we find that the angular momentum of the particle is a constant of motion which is a characteristic of motion under central force.

4. Motion of a Linear Harmonic Oscillator

Consider a particle of mass m undergoing linear harmonic motion along the x-axis. Let us consider the origin to be the mean or the equilibrium position of the particle and measure all displacements of the particle from the origin.

If x be the displacement of the particle at any instant of time t, the kinetic energy of the particle is

$$T = \frac{1}{2}m\dot{x}^2 \tag{87}$$

If k be the restoring force per unit displacement acting on the particle then the potential energy of the particle when its displacement is x is

$$V = \frac{1}{2}kx^2 \tag{88}$$

Since the system is conservative, we have the Hamiltonian for the oscillator given by

$$H = T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \tag{89}$$

The Lagrangian for the oscillator is

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{90}$$

The momentum conjugate to the coordinate x is by definition

$$p^x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

NOTES

which gives

$$\dot{x} = \frac{p_x}{m} \tag{91}$$

Using Equation (91) in Equation (89) we may write the Hamiltonian of the oscillator as

$$H = \frac{1}{2}m\frac{p_x^2}{m^2} + \frac{1}{2}kx^2$$

or
$$H = \frac{p_x^2}{2m} + \frac{1}{2}kx^2 \tag{92}$$

The Hamilton's canonical equations describing the oscillator are then

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \text{ (same as Equation (91))} \tag{93}$$

and
$$\dot{p}_x = -\frac{\partial H}{\partial x} = -kx \tag{94}$$

From Equation (93) we get

$$p^x = m\dot{x} \text{ and hence } \dot{p}_x = m\ddot{x} \tag{95}$$

Substituting Equation (95) in Equation (94) we obtain

$$m\ddot{x} = -kx$$

or
$$\ddot{x} + \frac{k}{m}x = 0 \tag{96}$$

Equation (96) is the well known equation for a linear harmonic oscillator.

5. Motion of a Charged Particle in an Electromagnetic Field

Consider a particle of mass m having charge q projected with a velocity $\vec{v}$ into an electromagnetic field described by the vector potential $\vec{A}(\vec{r}, \vec{t})$ and scalar potential ϕ.

The Lagrangian of the particle is given by

$$L = \text{Kinetic energy } (T) - \text{Potential energy } (U)$$

$$= T - q\phi + \frac{q}{c}\vec{v}\cdot\vec{A}$$

or
$$L = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\vec{v}\cdot\vec{A}$$

If x, y, z be the coordinates of the particle at the instant of time t, we may write the Lagrangian as

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{c}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z) - q\phi \tag{97}$$

NOTES

The momenta conjugate to the coordinates x, y and z are

$$p^x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{q}{c}A_x \quad (98)$$

$$p^y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{q}{c}A_y \quad (99)$$

$$p^z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} + \frac{q}{c}A_z \quad (100)$$

By definition, the Hamiltonian of the particle is

$$H = p^x\dot{x} + p^y\dot{y} + p^z\dot{z} - L \quad (101)$$

Substituting for $\dot{x}$, $\dot{y}$, $\dot{z}$ from Equation (98), (99) and (100) respectively and for L from Equation (97) in Equation (101), we obtain

$$\begin{aligned} H = {} & \frac{p_x}{m}\left(p_x - \frac{q}{c}A_x\right) + \frac{p_y}{m}\left(p_y - \frac{q}{c}A_y\right) + \frac{p_z}{m}\left(p_z - \frac{q}{c}A_z\right) \\ & - \frac{1}{2}m\left[\left(\frac{p_x - \frac{q}{c}A_x}{m}\right)^2 + \left(\frac{p_y - \frac{q}{c}A_y}{m}\right)^2 + \left(\frac{p_z - \frac{q}{c}A_z}{m}\right)^2\right] \\ & - \frac{q}{c}\left[\frac{p_x - \frac{q}{c}A_x}{m}A_x + \frac{p_y - \frac{q}{c}A_y}{m}A_y + \frac{p_z - \frac{q}{c}A_z}{z}A_z\right] + q\phi \end{aligned}$$

Simplifying the above we get

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi \quad (102)$$

The canonical equations are

$$(a) \quad \vec{v} = \dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} = \frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right) \quad (103)$$

and

$$(b) \quad \dot{\vec{p}} = \vec{\nabla}H = -\frac{1}{m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)\cdot\left(\frac{-q}{c}\vec{A}\right) - q\vec{\nabla}\phi$$

$$\text{or} \quad \dot{\vec{p}} = \frac{q}{c}\vec{v}\cdot\vec{A} - q\vec{\nabla}\phi \quad (104)$$

Check Your Progress

6. Define the Routhian or Routh function.
7. How do we get the time period of oscillation of the compound pendulum?

3.5 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

NOTES

1. The Hamiltonian of a system is defined to be the sum of the kinetic and potential energies expressed as a function of positions and their conjugate momenta.
2. Hamilton's variational principle, or the Principle of Least Action, is stated as

$$\delta S = \delta\int_{t_1}^{t_2} L \;\; q_k, \dot{q}_k, t \;\; dt = 0$$

3. Euler's formula for any arbitrary homogeneous function $f = f(x^1, x^2, ..., x^N)$ of m^{th} degree is

$$\sum_{l=1}^{N} \frac{\partial f}{\partial x_l} x_l = mf$$

4. The cyclic coordinates are the generalized coordinates of a certain physical system that do not occur explicitly in the expression of the characteristic function of these systems.
5. Ignorable coordinate is a generalized coordinate of a mechanical system that does not appear in the systems of Lagrangian function or other characteristic functions.
6. The Routhian or Routh function usually denoted by R is defined as

$$R = \;\; p\dot{q} - L$$

7. We get the time period of oscillation of the compound pendulum to be given by

$$T = \;\; 2\pi\sqrt{\frac{I}{mgl}}$$

3.6 SUMMARY

- The momentum of a particle p_i is defined in terms of its velocity r_i by, $p_i = m_i r_i$
- The kinetic energy is a homogeneous quadratic function of the generalized velocities for systems in which constraints involved are scleronomous, i.e., independent of time.
- For a conservative system in which constraints are independent of time, the Hamiltonian function is equal to the total energy E of the system.
- The Hamiltonian function which is equal to the total energy is a constant of motion for a system if it or the Lagrangian of the system does not depend on time explicitly.
- The Hamiltonian formulation of mechanics is superior to the Lagrangian formulation, mainly because of the identification of Hamiltonian function as representing the total energy of the system.

NOTES

3.7 KEY WORDS

- **Hamiltonian:** The Hamiltonian of a system is defined to be the sum of the kinetic and potential energies expressed as a function of positions and their conjugate momenta.
- **Ignorable coordinate:** It is a generalized coordinate of a mechanical system that does not appear in the systems of Lagrangian function or other characteristic functions.
- **Central force:** A central force on an object is a force that is directed along the line joining the object and the origin
- **Simple pendulum:** Simple pendulum is one which can be considered to be a point mass suspended from a string or rod of negligible mass.

3.8 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Deduce canonical equations from variational principle.
2. Describe the general conservation theorem briefly.
3. Show superiority of Hamiltonian formulation over Lagrangian formulation.

Long-Answer Questions

1. Explain Hamiltonian equations.
2. Give a detailed account of the ignorable coordinates.
3. Discuss the Routhian function.
4. Explain the applications of Hamiltonian dynamics.

3.9 FURTHER READINGS

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NOTES

BLOCK-II
HAMILTONIAN METHODS

UNIT 4 HAMILTON'S PRINCIPLE

Structure

4.0 INTRODUCTION

William Rowan Hamilton formulated the principle of stationary action which expresses that the dynamics of a physical system is established by a variational problem for a functional based on a single function, the Lagrangian, which holds all physical information relating the system and the forces acting on it. Hamilton's principle is applicable to the electromagnetic and gravitational fields also. It contributes in quantum mechanics, quantum field theory and criticality theories significantly. In this unit you will study Hamiltonian of the system and the Legendre transformation relations for the change of basis. Hamilton's principle for conservative system and principle of least action is also discussed.

4.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain Hamiltonian of the system and the method of Legendre transformation
- Describe Hamilton's principle for a conservative system
- Understand principle of least action and Lagrangian formulation of mechanics

4.2 HAMILTON'S PRINCIPLE: AN INTRODUCTION

NOTES

Legendre transformation refers to the mathematical method for changing the basis of the description of a system from one set of independent variables to another set of independent variables.

Consider a function $f = f(x, y)$ of two independent variables x and y.

The total differential of f is

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \tag{1}$$

Let us define
$$u = \frac{\partial f}{\partial x} \tag{2}$$

and
$$v = \frac{\partial f}{\partial y} \tag{3}$$

We may then write Equation (1) as

$$df = udx + vdy \tag{4}$$

Let us now consider u to be an independent variable and x a dependent variable in order to change our basis from the variables (x, y) to the variables (u, y).

Let $f' = f'(u, y)$ be a function of u and y defined according to

$$f' = f - ux \tag{5}$$

We then have
$$df' = df - udx - xdu \tag{6}$$

Using Equation (4) in Equation (6), we obtain

$$df' = udx + vdy - udx - xdu$$

or
$$df' = vdy - xdu \tag{7}$$

Since $f' = f'(u, y)$, we have

$$df' = \frac{\partial f'}{\partial u}du + \frac{\partial f'}{\partial y}dy \tag{8}$$

Comparing Equation (7) and (8) we obtain

$$x = -\frac{\partial f'}{\partial u} \tag{9}$$

and
$$v = \frac{\partial f'}{\partial y} \tag{10}$$

The relations given by Equation (9) and (10) are called the Legendre transformation relations for the change of basis from (*x, y*) to (*u, y*). It is possible to extend the above method if we need to transform more than one variable.

NOTES

Consider a mechanical system having *s* degrees of freedom. Let $q_1, \ldots, q_s$ be the generalized coordinates that describe the system.

In the Lagrangian formulation of mechanics, the independent variables are the *s* generalized coordinates and time. The Lagrangian function *L* that characterizes the system is, in general, a function of the generalized coordinates, the generalized velocities and time, i.e.,

$$L = L(q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s, t) = L(q, \dot{q}, t) \tag{11}$$

In Equation (11), q stands for all the coordinates and $\dot{q}$ stands for all the velocities.

We note that although the generalized velocities appear in the expression for *L* explicitly, they cannot be treated as independent variables because of being equal to the total time derivatives of the generalized coordinates. Hamilton developed an alternative formulation of mechanics by considering the independent variables for the system as the generalized co-ordinates $(q_1, \ldots, q_s)$, the generalized momenta $p_1, \ldots, p_s$ and time *t*. In this formulation, the generalized velocities are dependent functions such as

$$\dot{q}_k = \dot{q}_k(q_1, \ldots, q_s, p_1, \ldots, p_s, t) \tag{12}$$

We may note that the generalized momenta are derived variables defined in terms of the Lagrangian *L* as

$$p_k = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k} \tag{13}$$

It follows from the above that to go over from the Lagrangian formulation to the Hamiltonian formulation, we need to change our basis of description of the system from $(q, \dot{q}, t)$ set to the (q, p, t) set. Such a change of basis can be carried out by the method of Legendre transformation discussed in the previous section.

A new function $H = H(q_1, \ldots, q_s, p_1, \ldots, p_s, t) = H(q, p, t)$, which also characterizes the system under consideration, is defined in terms of the Lagrangian function of the system $L(q, \dot{q}, t)$ in a manner analogous to Equation (8) as

$$H(q, p, t) = \sum_{k=1}^{s} p_k \dot{q}_k - L(q, \dot{q}, t) \tag{14}$$

The function $H(q, p, t)$ given by Equation (14) is known as the Hamiltonian of the system.

NOTES

Check Your Progress

1. What do you understand by Legendre transformation?
2. Write the equations for the Legendre transformation relations for the change of basis.
3. What is the Lagrangian function?
4. Give the function which is known as the Hamiltonian of the system.

4.3 HAMILTON'S PRINCIPLE FOR A CONSERVATIVE SYSTEM – PRINCIPLE OF LEAST ACTION

Lagrangian formulation of mechanics, which is an alternative to Newtonian formulation, is based on one of the fundamental variational principles given by Hamilton known as the Hamilton's variational principle. It is important to note that the principle is stated in a form which is independent of any coordinate system and as such the principle can be used for dealing with non-mechanical systems and fields as well.

According to Hamilton, every mechanical system possesses a characteristic function of coordinates, velocities and time called the Lagrangian of the system usually denoted by the symbol *L*. If for a dynamical system having *s*-degrees of freedom, q_1, ….., q_s and $\dot{q}_1, \ldots.., \dot{q}_s$ be respectively the generalized coordinates and generalized velocities (both the coordinates and the velocities may be implicit as well as explicit functions of time), then the Lagrangian of the system is given by

$$L = L(q_1, \ldots.., q_s, \dot{q}_1, \ldots.., \dot{q}_s, t) \tag{15}$$

At any instant of time *t*, the configuration of the system can be represented by a point called the system point in the *s*-dimensional mathematical space, namely, the configuration space of the system. As time passes, the system point moves in the configuration space and traces out a definite curve or path during a definite interval of time. Hamilton's principle is concerned with the trajectory or the path which is followed by the system point.

The principle states that of all possible paths along which the system may move from one point to another in its configuration space between two given time instants, say t1 and t2, which are consistent with the constraints imposed on the system, if any, the actual path which the system follows is the one for which the time integral of the Lagrangian of the system is an extremum (either maximum or minimum).

NOTES

Mathematically, the principle is stated as

$$S = \int_{t_1}^{t_2} L dt = \text{An extremum} \qquad (16)$$

The line integral $\int_{t_1}^{t_2} L dt$ which has been denoted above by the symbol S, is called the Hamilton's principle function, or action integral, or simply the action during the time interval from t_1 to t_2.

In most of the dynamical problems, the minimum condition for the action S is satisfied. For this reason, the principle is also called Hamilton's principle of least action.

In terms of calculus of variation we can express Hamilton's principle given by Equation (16) as

$$\delta S = \delta \int_{t_1}^{t_2} L dt = 0$$

or

$$\delta S = \delta \int_{t_1}^{t_2} L\left(q_1, ..., q_s, \dot{q}_1, ..., \dot{q}_s, t\right) = 0 \qquad (17)$$

Check Your Progress

5. Give mathematical statement for Hamilton's principle for a conservative system.
6. What do you understand by Hamilton's principle for a conservative system?
7. Why Hamilton's principle for a conservative system is also called Hamilton's principle of least action?

4.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Legendre transformation refers to the mathematical method for changing the basis of the description of a system from one set of independent variables to another set of independent variables.
2. The relations given by the following equations are called the Legendre transformation relations for the change of basis from (x, y) to (u, y):

$$x = -\frac{\partial f'}{\partial u} \quad \text{and} \quad v = \frac{\partial f'}{\partial y}$$

NOTES

3. The Lagrangian function L that characterizes the system is, in general, a function of the generalized coordinates, the generalized velocities and time, i.e.,

$$L= L(q_1, ..., q_s, \dot{q}_1, ..., \dot{q}_s, t) = L(q, \dot{q}, t)$$

4. The function $H(q, p, t)$ given by the following equation is known as the Hamiltonian of the system.

$$H(q, p, t) = \sum_{k=1}^{s} p_k \dot{q}_k - L(q, \dot{q}, t)$$

5. Mathematically, the principle is stated as

$$S = \int_{t_1}^{t_2} Ldt = \text{An extremum}$$

The line integral $\int_{t_1}^{t_2} Ldt$ which has been denoted above by the symbol S, is called the Hamilton's principle function, or action integral, or simply the action during the time interval from t_1 to t_2.

6. The principle states that of all possible paths along which the system may move from one point to another in its configuration space between two given time instants, say t1 and t2, which are consistent with the constraints imposed on the system, if any, the actual path which the system follows is the one for which the time integral of the Lagrangian of the system is an extremum (either maximum or minimum).

7. In most of the dynamical problems, the minimum condition for the action S is satisfied. For this reason, the principle is also called Hamilton's principle of least action.

4.5 SUMMARY

- Although the generalized velocities appear in the expression for L explicitly, they cannot be treated as independent variables because of being equal to the total time derivatives of the generalized coordinates.
- The generalized momenta are derived variables defined in terms of the Lagrangian L as

$$p_k = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}_k}$$

- Every mechanical system possesses a characteristic function of coordinates, velocities and time called the Lagrangian of the system usually denoted by the symbol L.
- Hamilton's principle is concerned with the trajectory or the path which is followed by the system point.

4.6 KEY WORDS

- **Legendre transformation:** It refers to the mathematical method for changing the basis of the description of a system from one set of independent variables to another set of independent variables.
- **Mechanics:** The branch of science which deals with motion of objects under the action of forces, including the particular case in which a body continues at rest is called mechanics.
- **Mechanical system:** It is a system that manages power to complete a work that involves forces and movement. Here power means the rate of doing work or transferring heat.
- **Generalized coordinates:** The coordinates in a state space that together absolutely describe a system are called generalized coordinates. If they are selected so as to be independent of each other, the number of independent generalized coordinates matches the number of degrees of freedom of the system.

4.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Derive the Legendre transformation relations for the change of basis.
2. Describe Hamilton's principle briefly.

Long-Answer Questions

1. Discuss Hamilton's principle for a conservative system. Also state why it is called the principle of least action.
2. Give an introduction to Hamilton's principle and derive the Hamiltonian of the system.

4.8 FURTHER READINGS

NOTES

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UNIT 5 HAMILTON-JACOBI THEORY

NOTES

Structure

5.0 INTRODUCTION

The Hamilton-Jacobi equation named for William Rowan Hamilton and Carl Gustav Jacob Jacobi, explains extremal geometry in generalizations of problems from the calculus of variations. It is an alternative formulation of classical mechanics, equivalent to other formulations such as Newton's laws of motion, Lagrangian mechanics and Hamiltonian mechanics. It is a first-order partial non-linear differential equation, especially applicable in understanding conserved quantities for mechanical systems, which may be possible even when the mechanical problem itself cannot be solved completely. In this unit you will study characteristic function and Hamilton-Jacobi equation. You will learn Hamilton's principal function, Abbreviated action and solution of mechanical problems using Hamilton-Jacobi method. Application of action angle variable and Jacobi identity is also explained.

5.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain characteristic function and Hamilton-Jacobi equation
- Understand Hamilton's principal function and Hamilton's principal function
- Solve mechanical problems using Hamilton-Jacobi method and calculate motion of a body falling freely under gravity

- Describe action and angle variables, application of action angle variable to obtain the frequency of a linear harmonic oscillator
- Define Jacobi's identity

NOTES

5.2 CHARACTERISTIC FUNCTION AND HAMILTON-JACOBI EQUATION

Consider a mechanical system of s degrees of freedom described by the generalized coordinates $q_1, \ldots.., q_s$ and generalized momenta $p_1, \ldots.., p_s$. The Hamiltonian for the system which involves time explicitly is

$$H = H(q, p, t) = H(q_1, \ldots.., q_s, p_1, \ldots.., p_s, t) \tag{1}$$

The Hamilton's canonical equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k}; \quad \text{and} \quad \dot{q}_k = -\frac{\partial H}{\partial q_k} \tag{2}$$

Let us affect a canonical transformation from the old set of variables $\{q, p\}$ to a new set $\{Q, P\}$. Let F be the generating function of the transformation. If K represents the transformed Hamiltonian then the following equations hold

$$\dot{Q}_k = \frac{\partial K}{\partial P_k}\ ; \ \dot{P}_k = -\frac{\partial K}{\partial Q_k} \tag{3}$$

and
$$K = H + \frac{\partial F}{\partial t} \tag{4}$$

Let us consider the above transformation to be such that the new variables are constants of motion (Q_k = Constant, P_k = Constant), so that

$$\dot{Q}_k = 0 \ \text{ and } \ \dot{P}_k = 0 \tag{5}$$

According to Equation (3) and (5), we then get

$$\frac{\partial K}{\partial P_k} = 0 \ \text{ and } \ \frac{\partial K}{\partial Q_k} = 0 \tag{6}$$

Equation (6) allows us to take the transformed Hamiltonian K identically equal to zero. Equation (4) then demands the generating function F to be such that

$$H\ \ q, p, t\ + \frac{\partial F}{\partial t} = 0 \tag{7}$$

Let us choose the generating function F to be a function of old coordinates, new constant momenta and time, i.e.,

$$F = F(q, P, t) \tag{8}$$

From Equation $F_1(q, q, t) = F_2(q, p, t) - \Sigma P_k\, Q_k$, we then have

$$p_k = \frac{\partial F}{\partial q_k} \tag{9}$$

In view of Equation (9) we may rewrite Equation (7) as

$$H\left(q_1, ..., q_s, \frac{\partial F}{\partial q_1}, ..., \frac{\partial F}{\partial q_s}, t\right) + \frac{\partial F}{\partial t} = 0 \quad (10)$$

Equation (10) is referred to as the Hamilton–Jacobi equation. We may note that the Hamilton–Jacobi equation is a first-order partial differential equation in $(s + 1)$ variables, namely, q_1, ..., q_s, t.

By convention, solution of Equation (10) is denoted by S and is called the **Hamilton's Principal Function.**

The complete solution of Equation (10) involves $(s + 1)$ constants of integration. We may note that if we add a constant, say α, to the solution S, so that the solution may be written as $S + \alpha$ then we find that such a replacement of the solution satisfies Equation (10). Hence, we find that out of $(s + 1)$ constants mentioned above, one is an additive constant. We may thus write the general solution of the Hamilton–Jacobi equation as

$$S = S(q_1, ..., q_s, \alpha_1, ..., \alpha_s, t) + \alpha \quad (11)$$

In the above, α_1, α_2, ..., α_s are the s independent, non-additive constants of integration. Clearly, solution of the Hamilton–Jacobi equation S is a function of s coordinates, s constants and time. This is precisely the same description as that of the generating function F considered above. The constants can be chosen as the new momenta which are constants of motion. Thus,

$$\alpha_k = p_k;\ (k = 1, ..., s) \quad (12)$$

The new momenta which are constants can conveniently be chosen to be the momentum values p_{01}, p_{02}, ..., p_{0s} at the initial time $t = t_0$.

Transformation equations given by Equation (9) can be rewritten as

$$p_k = \frac{\partial S}{\partial q_k} \quad (13)$$

Equation (13) can be used to determine the relations between α_k, p_k and q_k at $t = t_0$.

We also have the transformation equation

$$Q_k = \frac{\partial S}{\partial P_k} = \frac{\partial S}{\partial \alpha_k} = \beta_k \text{ (say)}; \qquad k = 1, ..., s \quad (14)$$

We may choose the values of β_k as the momenta and they can also be expressed in terms of the initial values of the coordinates, namely, q_{10}, ..., q_{50}.

We may write Equation (14) also as

$$q_k = q_k(\alpha_k, \beta_k, t) \quad (15)$$

Equation (15) along with the relations

$$p_k = p_k(\alpha_k, \beta_k, t) \quad (16)$$

gives the solutions of the problem.

NOTES

Identification of the Solution of Hamilton–Jacobi Equation

From Equation (8) we get

$$\frac{dF}{dt} = \sum \frac{\partial F}{\partial q_k}\dot{q}_k + \sum \frac{\partial F}{\partial p_k}\dot{p}_k + \frac{\partial F}{\partial t}$$

Using Equation (5) and (9) in the above, we obtain

$$\frac{dF}{dt} = \sum p_k \dot{q}_k + \frac{\partial F}{\partial t} \tag{17}$$

Equation (7) when used in Equation (17) gives

$$\frac{dF}{dt} = \sum p_k \dot{q}_k - H(q, p, t)$$

or

$$\frac{dF}{dt} = L(q, q, t) \tag{18}$$

where $L\left(q, \dot{q}, t\right) = \sum p_k \dot{q}_k - H(q, p, t)$, is the Lagrangian of the system.

Integration of Equation (18) over time leads to

$$\int \frac{dF}{dt} dt = \int L\left(q, \dot{q}, t\right)$$

or

$$F = S + \text{constant} \tag{19}$$

In the above, S represents the familiar action of the system. We thus identify the solution of the Hamilton–Jacobi equation, i.e., the Hamilton's principal function as the indefinite time integral of the Lagrangian of the system, plus a constant.

We may now summarize the method for solving a mechanical problem using Hamilton–Jacobi method. The steps are:

(*i*) Construct the Hamiltonian H for the system in terms of old coordinates q, old momenta p and time t.

(*ii*) Write the Hamilton–Jacobi equation and find its complete integral S, i.e., the Hamilton's principal function.

(*iii*) Differentiate S with respect to s arbitrary constants α_k and equate the derivatives to the new constants β_k to obtain s algebraic equations, such as

$$\frac{\partial S}{\partial \alpha_k} = \beta_k \tag{20}$$

(*iv*) Solve the s equations thus obtained to get the old coordinates q_k as functions of time t and the $2s$ arbitrary constants

$$q_k = q_k(\alpha_1, \alpha_2, ..., \alpha_s, \beta_1, \beta_2, ..., \beta_s, t) \tag{21}$$

(*v*) Obtain the old momenta according to

$$p_k = \frac{\partial S}{\partial q_k} \tag{22}$$

(*vi*) Use Equation (21) and their time derivatives to determine 2*s* initial conditions of the given mechanical system.

Abbreviated Action, or Hamilton's Characteristic Function

NOTES

Consider a conservative mechanical system. The Hamiltonian of the system then has no explicit dependence on time and is a constant of motion representing the total energy *E* of the system. We have

$$S = \int_1^2 L dt = \int_1^2 \left[\sum p_k dq_k - H dt\right]$$

$$= \int_1^2 \sum p_k dq_k - \int_1^2 H dt$$

or $$S = S_0 - E(t_2 - t_1) \tag{23}$$

where $$S_0 = \int_1^2 \sum p_k dq_k \tag{24}$$

Substituting Equation (23) in the Hamilton–Jacobi equation given by Equation (10), we obtain

$$-E + H\left(q_1, \ldots, q_s, \frac{\partial S_o}{\partial q_1}, \ldots, \frac{\partial S_o}{\partial q_s}\right) = 0 \tag{25}$$

The solution S0 of the time-independent partial differential Equation (25) is called abbreviated action, or Hamilton's characteristic function.

5.2.1 Solution of Mechanical Problems Using Hamilton–Jacobi Method

A. One-dimensional Harmonic Oscillator Problem

Consider a one-dimensional harmonic oscillator of mass *m*. Let *q* and *p* be respectively the coordinate and momentum of the oscillator at any instant of time *t*.

The Hamiltonian function *H* of the oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \tag{26}$$

where $\omega = \sqrt{\frac{k}{m}}$, k being the force constant of the oscillator.

The Hamilton–Jacobi equation is given by

$$H(q, p) + \frac{\partial S}{\partial t} = 0$$

Using $p = \frac{\partial S}{\partial q}$, the above equation becomes

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2} m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0 \tag{27}$$

NOTES

We note that in the above equation, the only term that involves an explicit dependence of S on t is the term $\frac{\partial S}{\partial t}$. We may write the solution of Equation (27) in the form

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t \tag{28}$$

where α is a constant of integration.

From Equation (28), we get

$$\frac{\partial S}{\partial q} = \frac{\partial W}{\partial q}$$

and

$$\frac{\partial S}{\partial t} = -\alpha$$

Substituting the above in Equation (27), we get

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 = \alpha$$

or

$$\left(\frac{\partial W}{\partial q}\right)^2 = 2m\alpha - 2m \times \frac{1}{2}m\omega^2 q^2$$

or

$$\frac{\partial W}{\partial q} = \sqrt{2m\left(\alpha - \frac{1}{2}m\omega^2 q^2\right)} = m\omega\sqrt{\left(\frac{2\alpha}{m\omega^2} - q^2\right)}$$

Integrating the above gives

$$W = m\omega\int\sqrt{\left(\frac{2\alpha}{m\omega^2} - q^2\right)}dq \tag{29}$$

Using Equation (29), the solution given by Equation (28) takes the form

$$S = m\omega\int\sqrt{\left(\frac{2\alpha}{m\omega^2} - q^2\right)}dq - \alpha t \tag{30}$$

We have the transformation equations

$$P = \frac{\partial S}{\partial q} \quad \text{and} \quad \beta = \frac{\partial S}{\partial \alpha} \tag{31}$$

Use of Equation (30) yields

$$p = m\omega\left(\frac{2\alpha}{m\omega^2} - q^2\right)^{\frac{1}{2}} \tag{32}$$

and

$$\beta = \frac{1}{\omega}\int\frac{dq}{\sqrt{\frac{2\alpha}{m\omega^2} - q^2}} - t$$

or

$$\beta + t = -\frac{1}{\omega}\cos^{-1} q\sqrt{\frac{m\omega^2}{2\alpha}}$$

The above gives

$$q = \sqrt{\frac{2\alpha}{m\omega^2}} \cos\omega(t+\beta) \qquad (33)$$

The constants α and β in Equation (33) may be related to the initial values q_0 and p_0 of coordinate and momentum of the oscillator, respectively. Let at $t = 0$, the particle be at rest at a position displaced by q_0 from the equilibrium position. We then get from Equation (27)

$$\left(\frac{\partial S}{\partial q}\right)_o = p_o = \sqrt{2m}\left(\alpha - \frac{m\omega^2 q_o^2}{2}\right)^{\frac{1}{2}}$$

Since $p_0 = 0$, the above result gives

$$\alpha = \frac{m\omega^2 q_o^2}{2} \qquad (34)$$

From Equation (26) we find that α given by Equation (34) is the initial total energy of the system. Equation (34) gives

$$q_0 = \sqrt{\frac{2\alpha}{m\omega^2}} \qquad (35)$$

Using Equation (35) in Equation (33), we get

$$q = q_0 \cos \omega(t + \beta) \qquad (36)$$

Further, since $q = q_0$ at $t = 0$, according to Equation (36), we get

$$\cos \beta = 1 \text{ or } \beta = 0$$

We thus get

$$q = q_0 \cos \omega t \qquad (37)$$

Conclusion

The Hamilton's principal function S affects a contact transformation which transforms the oscillator with a canonical momentum $\alpha = H =$ Constant total energy and a coordinate β which is zero at $t = 0$. We may write Hamilton's principal function of the oscillator as

$$S = m\omega\int\sqrt{q_o^2 - q^2}\,dq - \frac{m\omega^2 q_o^2 t}{2}$$

[by using Equation (34) in Equation (30)]

Using the value of q from Equation (36), the above gives

$$S = m\omega^2 q_o^2 \int\left(\sin^2 \omega t - \frac{1}{2}\right)dt \qquad (38)$$

It is now easy to show that S given by Equation (38) is the time integral of the Lagrangian (L) of the oscillator. We have

$$L = \text{Kinetic energy} - \text{Potential energy}$$

NOTES

$$= \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2$$

Substituting from Equation (36), the above gives

$$L = \frac{m\omega^2 q_o^2}{2}\left[\sin^2 \omega t - \cos^2 \omega t\right]$$

$$= \frac{m\omega^2 q_o^2}{2}\left[\sin^2 \omega t - 1 + \sin^2 \omega t\right]$$

or
$$L = \frac{m\omega^2 q_o^2}{2}\left[2\sin^2 \omega t - 1\right]$$

Clearly,
$$\int L dt = m\omega^2 q_o^2 \int\left(\sin^2 \omega t - \frac{1}{2}\right)dt$$

B. Motion of a Body Falling Freely under Gravity

Consider a body of mass m falling freely under gravity. At some instant of time t, let v be the velocity of the body and z be its height above the ground. The kinetic energy of the body is

$$T = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

(p being the linear momentum of the body)

The potential energy (gravitational potential energy) is

$$V = mgz$$

The Hamiltonian which represents the total energy E of the body is thus

$$H = \frac{p^2}{2m} + mgz = E \tag{39}$$

We may write H in the usual notation of S as

$$H = \frac{1}{2m}\left(\frac{\partial S}{\partial z}\right)^2 + mgz \tag{40}$$

where S is the Hamilton's principle function.

Hamilton–Jacobi equation is given by

$$H + \frac{\partial S}{\partial t} = 0$$

Using Equation (40), the above gives

$$\frac{1}{2m}\left(\frac{\partial S}{\partial z}\right)^2 + mgz + \frac{\partial S}{\partial t} = 0 \tag{41}$$

The general solution of Equation (41) can be written as

$$S(z, \alpha, t) = W(z, \alpha) - \alpha t \tag{42}$$

NOTES

From the above, we get

$$\begin{pmatrix} \frac{\partial S}{\partial z} = \frac{\partial W}{\partial z} \\ \frac{\partial S}{\partial t} = -\alpha \end{pmatrix}$$

$$\frac{\partial S}{\partial t} = -\alpha \tag{43}$$

Using Equation (43), Equation (41) becomes

$$\frac{1}{2m}\left(\frac{\partial W}{\partial z}\right)^2 + mgz - \alpha = 0$$

or $$\left(\frac{\partial W}{\partial z}\right) = \left[2m \ \ \alpha - mgz\ \right]^{\frac{1}{2}}$$

Integrating over the variable z, we obtain

$$W = \int \left[2m(\alpha - mgz)\right]^{\frac{1}{2}} dz + A \tag{44}$$

where A is the constant of integration.

Using the result given by Equation (44) in Equation (42), we get

$$S = \int \left[2m(\alpha - mgz)\right]^{\frac{1}{2}} dz + C - \alpha t \tag{45}$$

We obtain from the above

$$\frac{\partial S}{\partial \alpha} = \frac{\sqrt{2m}}{2}\int \frac{dz}{\sqrt{\alpha - mgz}} - t = \beta$$

or $$\beta + t = \frac{\sqrt{2m}}{2}\frac{2\sqrt{\alpha - mgz}}{(-mg)} = -\sqrt{\frac{2}{m}}\frac{(\alpha - mgz)^{\frac{1}{2}}}{g}$$

or $$(\alpha - mgz)^{\frac{1}{2}} = -g\sqrt{\frac{m}{2}}(\beta + t)$$

or $$\alpha - mgz = g^2\frac{m}{2}(\beta + t)^2$$

or $$mgz = \alpha - \frac{mg^2}{2}(\beta + t)^2$$

or $$z = \frac{\alpha}{mg} - \frac{g}{2}(\beta + t)^2 \tag{46}$$

Let $z = z_0$ and $p = 0$, initially at $t = 0$.

Then we have

$$p = \frac{\partial W}{\partial z} = \sqrt{2m(\alpha - mgz_o)} = 0$$

The above gives

$$\alpha = mgz_0 \tag{47}$$

Using Equation (47) in Equation (46), we obtain

$$z = \frac{mgz_o}{mg} - \frac{g}{2}(\beta + t)^2$$

NOTES

or $$z = z_o - \frac{g}{2}(\beta + t)^2$$

Since at $t = 0$, $z = z_0$ we obtain

$$z_0 = z_o - \frac{g}{2}\beta^2$$

The above gives $\beta = 0$.

Hence, we obtain

$$z = z_o - \frac{1}{2}gt^2$$

This is the equation of motion for the freely falling body.

5.2.2 Illustrative Examples

Example 1: A single particle is moving under the Hamiltonian $H = \frac{p^2}{2}$.

(*a*) Solve the Hamilton-Jacobi equation for the generating function $s(q, d, t)$.

(*b*) If β and α are the transformed coordinate and momentum respectively, find the canonical transformation $q = q(\beta, \alpha)$ and $p = p(\beta, \alpha)$

Solution:

(*a*) We have the Hamilton-Jacobi equation as

$$\frac{\partial s}{\partial t} + H(q, p, t) = 0 \qquad (i)$$

where $$p = \frac{\partial s}{\partial q} \qquad (ii)$$

In the problem, $H = \frac{p^2}{2}$ and hence using Equation (*ii*) we obtain and according to Equation (*i*)

$$\frac{\partial s}{\partial t} + \frac{1}{2}\left(\frac{\partial s}{\partial q}\right)^2 = 0 \qquad (iii)$$

As H does not depend on q and t, the two terms on the left hand side of Equation (*iii*) can be set equal to, say, $-r$ and $+r$ where r is a function of p.

We then get

$$S = \sqrt{2\gamma}\; q - \gamma t \qquad (iv)$$

Setting $\alpha = \sqrt{2\gamma}$ we get the generating function

$$S = \alpha q - \frac{1}{2}\alpha^2 t \qquad (v)$$

(*b*) Considering the constant α to be the new momentum P we have the transformation equations

$$p = \frac{\partial s}{\partial q} = \alpha \qquad (vi)$$

$$Q = \frac{\partial s}{\partial P} = \frac{\partial s}{\partial \alpha} = q - \alpha t \qquad (vii)$$

NOTES

Example 2: The Hamiltonian of a system is given by

$$H = \frac{p^2}{2} - \frac{\mu}{q}$$

Solve the corresponding Hamilton-Jacobi equation.

Solution:

Given $$H = \frac{p^2}{2} - \frac{\mu}{q} \qquad (i)$$

The Hamilton-Jacobi equation for the system is

$$\frac{\partial s}{\partial t} + H(q, p, t) = 0 \qquad (ii)$$

We have $$p = \frac{\partial s}{\partial q}$$

So that using Equation (i) in Equation (ii) we get

$$\frac{\partial s}{\partial t} + \frac{1}{2}\left(\frac{\partial s}{\partial q}\right)^2 - \frac{\mu}{q} = 0 \qquad (iii)$$

We may set

$$S = f(t) + g(q) \qquad (iv)$$

where $f(t)$ is a function of only t while $g(q)$ is a function of only q.

Using Equation (iv), Equation (iii) becomes

$$\frac{df(t)}{dt} + \frac{1}{2}\left(\frac{dg(q)}{dq}\right)^2 - \frac{\mu}{q} = 0$$

or $$\frac{df(t)}{dt} = \frac{\mu}{q} - \frac{1}{2}\left(\frac{dg(q)}{dq}\right)^2 \qquad (v)$$

The L.H.S. of Equation (v) depends only on time while the R.H.S. depends only on q. Hence, we may set both sides equal to a constant say $\frac{\mu}{\alpha}$. Thus we get

$$\frac{df(t)}{dt} = \frac{\mu}{\alpha}$$

Integrating, we get

$$f(t) = \frac{\mu}{\alpha}$$

Also $$\frac{\mu}{q} - \frac{1}{2}\left(\frac{dg(q)}{dq}\right)^2 = \frac{\mu}{\alpha}$$

or $$\frac{1}{2}\left(\frac{dg(q)}{dq}\right)^2 = \frac{\mu}{q} - \frac{\mu}{\alpha}$$

NOTES

or $$\frac{dg(q)}{dq} = \sqrt{2\left(\frac{\mu}{q} - \frac{\mu}{\alpha}\right)}$$

Carrying out the integration we get

$$g(q) = \sqrt{2\mu\alpha}\ \text{arc}\ \sin\left[\sqrt{\frac{q}{\alpha}} + \left(\frac{2\mu q(\alpha - q)}{\alpha}\right)^{1/2}\right]$$

Then the solution of the Hamilton-Jacobi equation is

$$S = \frac{\mu}{\alpha}t + \sqrt{2\mu^2}\ \text{arc}\ \sin\left[\sqrt{\frac{q}{\alpha}} + \left(\frac{2\mu q(\alpha - q)}{\alpha}\right)^{1/2}\right]$$

Example 3: Write down the Hamilton-Jacobi equation for a three-dimensional harmonic oscillator and obtain the solution of the equation.

Solution:

The Hamiltonian function of the three-dimensional oscillator can be written as

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}p_1 q_1^2 + \frac{1}{2}k_2 q_2^2 + \frac{1}{2}k_3 q_3^2 \qquad (i)$$

where we have assumed the spring constants k_1, k_2 and k_3 along the three cartesian axes (referred in the above by the suffixes 1, 2, 3) to be different.

Since the oscillator is a conservative system, the Hamiltonian H does not depend on time explicitly and instead it is constant of motion. Thus

$$H = E(\text{say}) \qquad (ii)$$

The Hamilton's principal function S is thus

$$S(q_j, P_j, t) = W(q_j, p_j) - Et \qquad (iii)$$

We have $$p_j = \frac{\partial W}{\partial q_j} \qquad (iv)$$

Using the above we may write the Hamiltonian given by Equation (i) as

$$H = \sum_{j=1}^{3}\left[\frac{1}{2m}\left(\frac{\partial W}{\partial q_j}\right)^2 + \frac{1}{2}k_j q_j^2\right] = E$$

The above can be alternatively written as

$$\sum_{j=1}^{3}\left[\left(\frac{\partial W}{\partial q_j}\right)^2 + mk_j \cdot q_j^2\right] = 2mE \qquad (v)$$

Equation (5) is the Hamilton-Jacobi equation for the oscillator. It can be solved using the method of separation of variables according to which we may write

$$W = W_1(q_1) + W_2(q_2) + W_3(q_3) \qquad (vi)$$

Substituting Equation (vi) in Equation (v) we get three equations.

$$\left(\frac{\partial W_1}{\partial q_1}\right)^2 + mk_1 q_1^2 = 2m\alpha_1$$

$$\left(\frac{\partial W_2}{\partial q_2}\right)^2 + mk_2 q_2^2 = 2m\alpha_2 \qquad (vii)$$

$$\left(\frac{\partial W_3}{\partial q_3}\right)^2 + mk_3 q_3^2 = 2m\alpha_3$$

In the above

$$\alpha_1 + \alpha_2 + \alpha_3 = E \qquad (viii)$$

On integrating Equation (*vii*) we obtain

$$\begin{aligned} W_1 &= \int (2m\alpha_1 - mk_1 q_1)^2 \, dq_1 \\ W_2 &= \int (2m\alpha_2 - mk_2 q_2^2) \, dq_2 \\ W_3 &= \int (2m\alpha_3 - mk_3 q_3^2)^2 \, dq_3 \end{aligned} \qquad (ix)$$

The constants α_1, α_2, α_3 are designated as the new momenta P_1, P_2, P_3 respectively. The new constant coordinates are given by

$$\begin{aligned} Q_1 &= \frac{\partial W_1}{\partial P_1} = \frac{\partial W_1}{\partial \alpha_1} = \sqrt{\frac{m}{2}} \int \frac{dq_1}{\sqrt{\alpha_1 - \frac{1}{2} k_1 q_1^2}} \\ Q_2 &= \frac{\partial W_2}{\partial P_2} = \frac{\partial W_2}{\partial \alpha_2} = \sqrt{\frac{m}{2}} \int \frac{dq_2}{\sqrt{\alpha_2 - \frac{1}{2} k_2 q_2^2}} \\ Q_3 &= \frac{\partial W_3}{\partial P_3} = \frac{\partial W_3}{\partial \alpha_3} = \sqrt{\frac{m}{2}} \int \frac{dq_3}{\sqrt{\alpha_3 - \frac{1}{2} k_3 q_3^2}} \end{aligned} \qquad (x)$$

For the conservative system

$$H = K = E = \alpha_1 + \alpha_2 + \alpha_3 \qquad (xi)$$

The equations of motion in the new coordinates are

$$\begin{aligned} \theta_1 &= \frac{\partial k}{\partial P_1} = \frac{\partial E}{\partial \alpha_1} = 1 \\ \theta_2 &= \frac{\partial k}{\partial P_2} = \frac{\partial E}{\partial \alpha_2} = 1 \\ \theta_2 &= \frac{\partial k}{\partial P_3} = \frac{\partial E}{\partial \alpha_3} = 1 \end{aligned} \qquad (xii)$$

The above equations on integration give

$$\begin{aligned} Q_1 &= t + \beta_1 \\ Q_2 &= t + \beta_2 \\ Q_3 &= t + \beta_3 \end{aligned} \qquad (xiii)$$

From Equation (*x*) and (*xiii*) we obtain

$$t + \beta_1 = \sqrt{\frac{m}{2}} \int \frac{dq_1}{\sqrt{\alpha_1 - \frac{1}{2} k_1 q_1^2}}$$

NOTES

or $$\sqrt{\frac{m}{2\alpha_1}}\int\frac{dq_1}{\sqrt{1-\frac{k_1q_1^2}{2\alpha_1}}} = t+\beta_1$$

or $$\sqrt{\frac{m}{k_1}}\sin^{-1} q_1\sqrt{\frac{k_1}{2\alpha_1}} = t+\beta_1$$

or $$q_1 = \sqrt{\frac{2\alpha_1}{mw_1^2}}\sin w_1(t+\beta_1) \quad (xiv)$$

Similarly, we obtain

$$q_2 = \sqrt{\frac{2\alpha_2}{mw_2^2}}\sin w_2(t+\beta_2) \quad (xv)$$

$$q_3 = \sqrt{\frac{2\alpha_3}{mw_3^2}}\sin w_3(t+\beta_3) \quad (xvi)$$

In the above

$$w_j = \sqrt{\frac{k_j}{m}},\ j = 1, 2, 3 \quad (xvii)$$

Example 4: Using Hamilton-Jacobi method discuss the motion of a particles of mass m moving in a uniform gravitational field along the z-axis.

Solution:

Let at any instant of time t, p_x, p_y, p_z be the components of momentum of the particle along the x, y and z axes, respectively. We then have the kinetic energy of the particle as.

$$T = \frac{1}{2m}(p_x^2+p_y^2+p_z^2)$$

If the z-coordinate of the particle at the instant t be z, is

$$V = mgz$$

Thus, the Hamiltonian of the particle is

$$H = \frac{1}{2m}(p_x^2+p_y^2+p_z^2)+mgz \quad (i)$$

The Hamilton-Jacobi equation for the particle is thus

$$\frac{1}{2m}\left[\left(\frac{\partial s_o}{\partial x}\right)^2+\left(\frac{\partial s_o}{\partial y}\right)^2+\left(\frac{\partial s_o}{\partial z}\right)^2\right]+mgz = E \quad (ii)$$

Let us set

$$s_0 = S_1(x)+S_2(y)+S_3(z) \quad (iii)$$

Substituting Equation (iii) in Equation (ii) we get

$$\frac{1}{2m}\left[\left(\frac{\partial s_1}{\partial x}\right)^2+\left(\frac{\partial s_2}{\partial y}\right)^2+\left(\frac{\partial s_3}{\partial z}\right)^2\right]+mgz = E$$

NOTES

or $$\frac{1}{2m}\left[\alpha_1^2+\alpha_2^2+\left(\frac{\partial s_3}{\partial z}\right)^2\right]+mgz = E \qquad (iv)$$

From (i) we find x and y to be the cyclic coordinates and hence

$$\frac{\partial s_1}{\partial x} = \alpha_1 = \text{Constant}$$

$$\frac{\partial s_2}{\partial y} = \alpha_2 = \text{Constant} \qquad (v)$$

We may write Equation (iv) as

$$\alpha_1^2+\alpha_2^2+\left(\frac{\partial s_3}{\partial z}\right)^2 = 2mE-2m^2gz$$

or $$\frac{\partial s_z}{\partial z} = \left[2mE-2m^2gz-\alpha_1^2-\alpha_2^2\right]^{1/2}$$

Integrating the above we get

$$s_3 = \int\left[2mE-2m^2g\,z-\alpha_1^2-d_2^2\right]^{1/2}dz \qquad (vi)$$

Integrating Equation (v) we obtain

$$s_1 = \int\alpha_1 dx,\ s_2 = \int\alpha_2 dy \qquad (vii)$$

Substituting Equation (vi) and (vii) in Equation (iii) we obtain

$$s_0 = \int\alpha_1 dx+\int\alpha_2 dy+\int[2m(E-mgz)-\alpha_1^2-\alpha_2^2]^{1/2}dz \quad (viii)$$

At $t=t_0, x_0, y_0, z_0$ be the coordinates of the particle, we get the Hamilton-Jacobi function for the particle as

$$s = s_0-E\int_{t_0}^{t}dt$$

or $$S=\int_{x_0}^{x}\alpha_1 dx+\int_{y_0}^{y}\alpha_2 dy+\int_{z_0}^{z}\left[2m(E-mgz)-\alpha_1^2-d_2^2\right]^{1/2}dz-E(t-t_0) \qquad (ix)$$

We now have

$$\beta_1 = \frac{\partial s}{\partial\alpha_1},\ \beta_2=\frac{\partial s}{\partial\alpha_2},\ \beta_3=\frac{\partial s}{\partial\alpha_3}=\frac{\partial s}{\partial E}$$

We thus get

$$\beta_1 = \frac{\partial}{\partial\alpha_1}\int_{x_0}^{x}\alpha_1 dx+\int_{z_0}^{z}\alpha_1[2m(E-mgz)-\alpha_1^2-\alpha_2^2]^{-1/2}dz$$

or $$\beta_1 = x-x_0-\int_{z_0}^{z}\alpha_1[2m(E-mgz)-\alpha_1^2-\alpha_2^2]^{-1/2}dz \qquad (x)$$

Similarly,

$$\beta_2 = y-y_0-\int_{z_0}^{z}\alpha_2[2m(E-mgz)-\alpha_1^2-\alpha_2^2]^{-1/2}dz \qquad (xi)$$

$$\beta_3 = t-t_0 = \int_{z_0}^{z}m[2m(E-mgz)-\alpha_1^2-\alpha_2^2]^{-1/2}dz \qquad (xii)$$

Using the initial values at $t = t_0$ we get

$$\beta_1 = 0 \qquad (xiii)$$

$$\beta_2 = 0 \qquad (xiv)$$

NOTES

We may hence write

$$x - x_0 = \frac{1}{m}\alpha_1(t - t_0) \qquad (xv)$$

$$y - y_0 = \frac{1}{m}\alpha_2(t - t_0) \qquad (xvi)$$

$$z - z_0 = \left[2m(E - mgz_0) - \alpha_1^2 - \alpha_2^2\right]^{1/2} \frac{t - t_0}{m} - \frac{g}{2}(t - t_0)^2 \quad (xvii)$$

Equation (*xvii*) can be written as

$$z = -\frac{g}{2}(t - t_0)^2 + \frac{1}{m}\left[2m(E - mgz_0)\alpha_1^2 - \alpha_2^2\right]^{1/2}(t - t_0) + z_0$$

The above shows that the *z*-coordinate of the particle varies with time in a parabolic manner.

Example 5: Solve the problem of the motion of particle of mass *m* moving under a central force using Hamilton-Jacobi method.

Solution:

We know that such a motion takes place in a plane and is hence a two-dimensional motion. The convenient generalized coordinates are the polar coordinates r and θ in terms of which the Hamiltonian of the particle is given by

$$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) + U(r) \qquad (i)$$

Where $U(r)$ is potential corresponding to the central force, p_r and p_q are the momenta congugate to the coordinates r and θ respectively. We may write

$$s = s_0 - Et \text{ (since the system is conservative)} \qquad (ii)$$

The Hamilton-Jacobi equation for the problem is then

$$\frac{1}{2m}\left(\frac{\partial s_0}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial s_0}{\partial \theta}\right)^2 + U(r) + U(r) - E = 0 \qquad (iii)$$

Using the method of separation of variables and noting that θ is cyclic, we can obtain s_0 by integrating Equation (*iii*). The s_0 thus obtained when substituted in Equation (*ii*) gives

$$s = p_\theta + \theta + \int [2m\{E - U(r)\} - \frac{p_\theta^2}{r^2}]^{1/2} d\theta - Et \qquad (iv)$$

For the problem we have

$$\alpha_1 = E \text{ and } \alpha_2 = p_q. \qquad (v)$$

$$\beta_1 = \frac{\partial s}{\partial \alpha_1} = \frac{\partial S}{\partial E} = \int m\left[2m\{E - U(r)\} - \frac{p_\theta^2}{r^2}\right]^{-1/2} dr - t$$

$$\beta_2 = \frac{\partial s}{\partial \alpha_2} = \frac{\partial s}{\partial p_\theta} = \theta - \int \frac{p_\theta}{r^2}\left[2m\{E - U(r)\} - \frac{p_\theta^2}{r^2}\right]^{-\frac{1}{2}} \quad (vi)$$

NOTES

Both β_1 and β_2 are constants. Hence we may write Equation (v) and (vi) alternatively as

$$t = \int m\left[2m\{E - U(r)\} - \frac{p_\theta^2}{r^2}\right]^{-1/2} + \text{constant} \quad (vii)$$

$$\theta = \int \frac{p_\theta}{r^2}\left[2m\{E - U(r)\} - \frac{p_\theta^2}{r^2}\right]^{-1/2} + \text{constant} \quad (viii)$$

The above equations give the dependence of r on t and θ and hence give the path of motion of the particle.

5.2.3 Action and Angle Variables

Let us consider a periodic system having one degree of freedom. Let q be the generalized coordinate and p the generalized momentum which describe the system. Considering the system to be conservative we have the Hamiltonian of the system given by

$$H = H(q, p) = \alpha(\text{constant}) \quad (48)$$

According to the above, the momentum p for the system is a function of q and the constant α, i.e.,

$$p = p\,(q, \alpha)$$

A knowledge of the above function gives the trajectory of the system in its two-dimensional phase space.

A practical example of such a conservative one-dimensional system is a linear harmonic oscillator for which p is given by Equation (32)

$$p = m\omega\left(\frac{2\alpha}{m\omega^2} - q^2\right)^{\frac{1}{2}}$$

or
$$p^2 = \frac{m^2\omega^2 2\alpha}{m\omega^2} - m^2\omega^2 q^2$$

or
$$p^2 = 2m\alpha - m^2\omega^2 q^2 \quad (49)$$

The above can be written as

$$\frac{p^2}{2m\alpha} + \frac{m^2\omega^2 q^2}{2m\alpha} = 1$$

or
$$\frac{p^2}{2m\alpha} + \frac{q^2}{\left(\frac{2\alpha}{m\omega^2}\right)} = 1 \quad (50)$$

Clearly, the representative point in the phase space of the oscillator traces an ellipse as shown in the Fig. 5.1.

NOTES

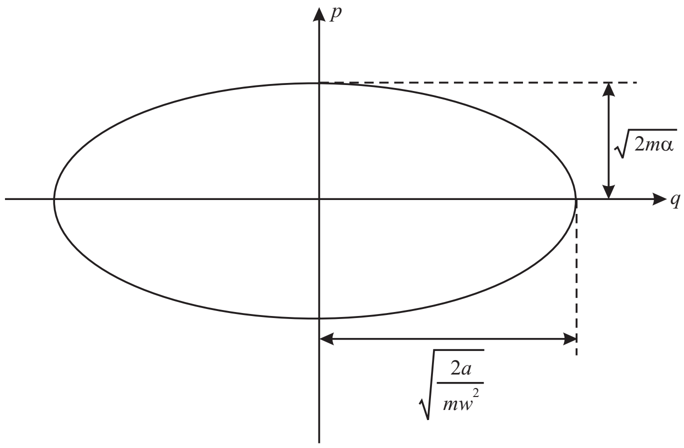

Fig. 5.1

We now introduce a new variable J for the oscillator according to

$$J = \oint p dq \tag{51}$$

where the integration is taken over one period round the ellipse. J has the dimensions of angular momentum (moment of linear momentum) and is called the phase integral or the action variable for the oscillator.

Substituting for p from Equation (49) in Equation (51) we get

$$J = \oint \sqrt{2m\alpha - m^2\omega^2 q^2}\, dq$$

The above yields J as a function of α (because q is integrated out) or the Hamiltonian H, i.e.,

$$J = J(\alpha) = J(H) \tag{52}$$

Alternatively, we obtain α or H as a function J

$$a = H = H(J) \tag{53}$$

The Hamilton's characteristic function S_0 can be written as

$$S_0 = S_0(q, J) \tag{54}$$

The generalized coordinate conjugate to J is called the angle variable and is defined by the transformation equation

$$w = \frac{\partial \delta_o}{\partial J} \tag{55}$$

The other transformation equation which gives p is

$$p = \frac{\partial \delta_o}{\partial q} \tag{56}$$

We thus get the equation of motion for the angle variable to be

$$\dot{w} = \frac{\partial H(J)}{\partial J} = \nu(J) \tag{57}$$

where ν (J) is a constant function of the action variable J only.

We may write the solution of Equation (57) as

$$w = vt + \beta \tag{58}$$

where β is a constant. We find the angle variable to vary linearly with time.

If Δw be the change in the variable w as q undergoes a change of one cycle, we get

$$\Delta w = \oint \frac{\partial W}{\partial q} dq$$

or
$$\Delta w = \oint \frac{\partial}{\partial q}\frac{\partial \delta_o}{\partial J} dq = \oint \frac{\partial^2 \delta_o}{\partial q\, \partial J} dq$$

or
$$\Delta w = \frac{d}{dJ}\oint \frac{\partial \delta_o}{\partial q} dq \quad (J \text{ being independent of } q) \tag{59}$$

Using Equation (56) in the above, we obtain

$$\Delta w = \frac{d}{dJ}\oint pdq = \frac{dJ}{dJ} = 1 \quad \text{(using Equation(51))} \tag{60}$$

If T be the time required for q to complete one cycle then according to Equation (58) we get

$$w = vt \tag{61}$$

In view of Equation (61), Equation (60) gives

$$vt = 1$$

or
$$T = \frac{1}{\nu} \tag{62}$$

It is clear from the above that since T represents the time period for one cycle q, v must represent the frequency, i.e., v gives the number of cycles of q in unit time.

Since v has the dimensions of frequency, according to Equation (58), w must have the dimension of angle.

We observe that use of action angle variables in periodic conservative mechanical systems provides simplified method of obtaining time period, or frequency of motion without requiring any detailed treatment.

5.2.4 Application of Action Angle Variable to Obtain the Frequency of a Linear Harmonic Oscillator

Consider the mechanical problem of a one-dimensional harmonic oscillator described in Sec. 6.3. The action variable J for the oscillator is given by

$$J = \oint pdq$$

Using p given by Equation (32), the above gives

$$J = \oint \sqrt{2m\alpha - m^2\omega^2 q^2}\, dq \tag{63}$$

where $\alpha = H = E =$ Total energy of the system and $\omega = \sqrt{\frac{k}{m}}$ being the force constant.

NOTES

To evaluate the integral in Equation (63), let us define a variable θ as

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\theta \tag{64}$$

From Equation (64) and (63) we get

$$J = \int_0^{2\pi} \sqrt{2m\alpha - m^2\omega^2\frac{2\alpha}{m\omega^2}\sin^2\theta} \times \sqrt{\frac{2\alpha}{m\omega^2}}\cos\theta\, d\theta$$

$$J = \int_0^{2\pi} \sqrt{2m\alpha\left(1-\sin^2\theta\right)}\sqrt{\frac{2\alpha}{m\omega^2}}\cos\theta\, d\theta$$

or $$= \int_0^{2\pi} \sqrt{2m\alpha\cos^2\theta\frac{2\alpha}{m\omega^2}}\cos\theta\, d\theta$$

$$= \int_0^{2\pi} \frac{2\alpha}{\omega}\cos^2\theta\, d\theta = \int_0^{2\pi} \frac{\alpha}{\omega}\left(1+\cos^2\theta\right)d\theta$$

or $$J = \frac{\alpha}{\omega}\left[2\pi + \left\{\frac{\sin^2\theta}{2}\right\}_0^{2\pi}\right]$$

or $$J = \frac{2\pi\alpha}{\omega} = \frac{2\pi E}{\omega} \tag{65}$$

The above gives

$$\alpha = H = \frac{\omega J}{2\pi} \tag{66}$$

Using Equation(57), we get the frequency of the oscillator to be

$$v = \frac{\partial H}{\partial J}$$

or $$v = \frac{\partial}{\partial J}\left(\frac{\omega J}{2\pi}\right) = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

5.2.5 Jacobi's Identity

If *f, g* and *k* are any three functions of the coordinates (*q*'s) and the momenta (*p*'s) describing a mechanical system, then the following relation holds

$$[f,[g, k]] + [g, [k, f]] + [k, [f, g]] = 0 \tag{67}$$

The relation expressed in Equation (67) is known as **Jacobi's identity**.

Proof: Using the definition and properties of Poisson bracket we get

$$[f, [g, k]] - [g, [f, k]] = \left[f, \sum_j \left(\frac{\partial g}{\partial q_j}\frac{\partial k}{\partial p_j} - \frac{\partial g}{\partial p_j}\frac{\partial k}{\partial q_j}\right)\right]$$

$$-\left[g, \sum_j \left(\frac{\partial f}{\partial q_j}\frac{\partial k}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial k}{\partial q_j}\right)\right] \tag{68}$$

Let us put

$$\left.\begin{aligned}\sum_j \frac{\partial g}{\partial q_j}\frac{\partial k}{\partial p_j} &= a\\ \sum_j \frac{\partial g}{\partial p_j}\frac{\partial k}{\partial q_j} &= b\\ \sum_j \frac{\partial f}{\partial q_j}\frac{\partial k}{\partial p_j} &= c\\ \sum_j \frac{\partial f}{\partial p_j}\frac{\partial k}{\partial q_j} &= d\end{aligned}\right\} \quad (69)$$

NOTES

Using Equation (69), we may write Equation (68) as

$$\begin{aligned}\left[f,[g,k]\right]-\left[g,[f,k]\right] &= [f,(a-b)]-[g,(c-d)]\\ &= [f,a]-[f,b]-[g,c]+[g,d]\\ &= \left[f,\sum_j\left(\frac{\partial g}{\partial q_j}\frac{\partial k}{\partial p_j}\right)\right]-\left[f,\sum_j\left(\frac{\partial g}{\partial p_j}\frac{\partial k}{\partial q_j}\right)\right]\\ &\quad -\left[g,\sum_j\left(\frac{\partial f}{\partial q_j}\frac{\partial k}{\partial p_j}\right)\right]+\left[g,\sum_j\left(\frac{\partial f}{\partial p_j}\frac{\partial k}{\partial q_j}\right)\right]\\ &= \left[f,\sum_j\frac{\partial g}{\partial q_j}\right]\sum_j\frac{\partial k}{\partial p_j}+\left[f,\sum_j\frac{\partial k}{\partial p_j}\right]\sum_j\frac{\partial g}{\partial q_j}\\ &\quad -\left[f,\sum_j\frac{\partial g}{\partial p_j}\right]\sum_j\frac{\partial k}{\partial q_j}-\left[f,\sum_j\frac{\partial k}{\partial q_j}\right]\sum_j\frac{\partial g}{\partial p_j}\\ &\quad -\left[g,\sum_j\frac{\partial f}{\partial q_j}\right]\sum_j\frac{\partial k}{\partial p_j}-\left[g,\sum_j\frac{\partial k}{\partial p_j}\right]\sum_j\frac{\partial f}{\partial q_j}\\ &\quad +\left[g,\sum_j\frac{\partial f}{\partial p_j}\right]\sum_j\frac{\partial k}{\partial q_j}+\left[g,\sum_j\frac{\partial k}{\partial q_j}\right]\sum_j\frac{\partial f}{\partial p_j}\end{aligned}$$

$$= \sum_i\left\{\frac{\partial k}{\partial q_j}\left(\left[\frac{\partial f}{\partial p_j},g\right]+\left[f,\frac{\partial g}{\partial p_j}\right]\right)+\frac{\partial k}{\partial p_j}\left(\left[\frac{\partial f}{\partial q_j},g\right]+\left[f,\frac{\partial g}{\partial q_j}\right]\right)\right\}$$

$$+ \sum_i\left\{\frac{\partial g}{\partial q_j}\left[f,\frac{\partial k}{\partial p_j}\right]-\frac{\partial g}{\partial p_j}\left[f,\frac{\partial k}{\partial q_j}\right]-\frac{\partial f}{\partial q_j}\left[g,\frac{\partial k}{\partial p_j}\right]+\frac{\partial f}{\partial p_j}\left[g,\frac{\partial k}{\partial q_j}\right]\right\}$$

(70)

$$\frac{\partial}{\partial x}[f,g] = \left[\frac{\partial f}{\partial x},g\right]+\left[f,\frac{\partial g}{\partial x}\right] \quad (71)$$

Using Equation(71) in Equation(70), we get

$$[f,[g,k]]-[g,[f,k]] = \sum_j\left\{-\frac{\partial k}{\partial q_j}\frac{\partial}{\partial p_j}[f,g]+\frac{\partial k}{\partial p_j}\frac{\partial}{\partial q_j}[f,g]\right\}$$

(the last four terms on the RHS of Equation (70) on expansion cancel each other out)

or $$\left[f,[g,k]\right]-\left[g,[f,k]\right]=-\left[k,[f,g]\right]$$

or $$\left[f,[g,k]\right]-\left[g,[f,k]\right]+\left[k,[f,g]\right]=0$$

or $$\left[f,[g,k]\right]+\left[g,[f,k]\right]+\left[k,[f,g]\right]=0$$

The above is the Jacobi's identity.

Jacobi's identity is helpful in finding the constants or integrals of motion for a system.

Check Your Progress

1. What is the Hamiltonian for the system which involves time explicitly?
2. What are the Hamilton's canonical equations?
3. Which equation is referred to as the Hamilton-Jacobi equation?
4. What will be the Hamiltonian function of the one-dimensional harmonic oscillator?
5. Write Hamilton's principal function of the oscillator.
6. Write the equation of motion for the freely falling body.
7. Define Jacobi's identity.

5.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Consider a mechanical system of s degrees of freedom described by the generalized coordinates $q_1, \ldots.., q_s$ and generalized momenta $p_1, \ldots..,$ p_s. The Hamiltonian for the system which involves time explicitly is
$$H = H(q, p, t) = H(q_1, \ldots.., q_s, p_1, \ldots.., p_s, t)$$
2. The Hamilton's canonical equations are
$$\dot{q}_k = \frac{\partial H}{\partial p_k}; \quad \text{and} \quad \dot{q}_k = -\frac{\partial H}{\partial q_k}$$
3. $$H\left(q_1, \ldots.., q_s, \frac{\partial F}{\partial q_1}, \ldots.., \frac{\partial F}{\partial q_s}, t\right) + \frac{\partial F}{\partial t} = 0$$
4. Consider a one-dimensional harmonic oscillator of mass m. Let q and p be respectively the coordinate and momentum of the oscillator at any instant of time t.

   The Hamiltonian function H of the oscillator is
$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$
5. We may write Hamilton's principal function of the oscillator as
$$S = m\omega\int\sqrt{q_o^2 - q^2}\,dq - \frac{m\omega^2 q_o^2 t}{2}$$

6. $z = z_o - \frac{1}{2}gt^2$

7. If f, g and k are any three functions of the coordinates (q's) and the momenta (p's) describing a mechanical system, then the following relation holds

$$[f,[g, k]] + [g, [k, f]] + [k, [f, g]] = 0$$

The relation expressed in Equation (67) is known as Jacobi's identity.

5.4 SUMMARY

- The Hamilton's canonical equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k}; \qquad \text{and} \qquad \dot{q}_k = -\frac{\partial H}{\partial q_k}$$

- The Hamilton–Jacobi equation is a first-order partial differential equation in $(s + 1)$ variables, namely, $q_1, \ldots\ldots, q_s, t$.
- We write the general solution of the Hamilton–Jacobi equation as

$$S = S(q_1, \ldots\ldots, q_s, \alpha_1, \ldots\ldots, \alpha_s, t) + \alpha$$

- The Hamiltonian of the system then has no explicit dependence on time and is a constant of motion representing the total energy E of the system.
- We may write Hamilton's principal function of the oscillator as

$$S = m\omega \int \sqrt{q_o^2 - q^2}\, dq - \frac{m\omega^2 q_o^2 t}{2}$$

5.5 KEY WORDS

- **Jacobi's identity:** It is a characteristic of a binary operation which explains how the order of evaluation influences the result of the operation.
- **Canonical coordinates:** The sets of coordinates on phase space which can be used to explain a physical system at any given point in time are called Canonical coordinates**.**
- **Action-angle coordinates:** They are a set of canonical coordinates useful in solving many integrable systems. The method of action-angles is useful for obtaining the frequencies of oscillatory motion without solving the equations of motion.
- **Poisson bracket:** It is an important binary operation in Hamiltonian mechanics which governs the time evolution of a Hamiltonian dynamical system.

5.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short-Answer Questions

1. Summarize the method for solving a mechanical problem using Hamilton-Jacobi method?
2. Describe the Hamilton-Jacobi equation briefly.
3. Write a short note on Hamilton's characteristic function.
4. Describe briefly application of action angle variable to obtain the frequency of a linear harmonic oscillator.

Long-Answer Questions

1. Describe Lagrange's equations for conservative systems.
2. Discuss Hamilton's principal function.
3. Give solution of one dimensional harmonic oscillator problem using Hamilton-Jacobi method.
4. Explain motion of a body falling freely under gravity.
5. Give a detailed account of action and angle variables.

5.7 FURTHER READINGS

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UNIT 6 CANONICAL TRANSFORMATIONS

NOTES

Structure

6.0 INTRODUCTION

A canonical transformation is a change of canonical coordinates that protects the form of Hamilton's equations. They constitute the basis for the Hamilton-Jacobi equations. The class of canonical transformations is much extensive, as the old generalized coordinates, momenta and even time may be combined to form the new generalized coordinates and momenta. In this unit you will study phase space and Liouville's theorem. You will learn special transformations with examples of physically meaningful canonical transformations and conditions for canonical transformation. You will examine Lagrange brackets and Poisson bracket. Calculation of variations is discussed at the end of this unit.

NOTES

6.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand phase space, configuration space, hodograph, and Liouville's theorem
- Describe special transformations explaining generating function for canonical transformation
- Interpret transformation relations for different forms of the generating function and infinitesimal transformation
- Analyse Lagrange brackets and Poisson brackets
- Grasp constants or integrals of motion and canonical transformation and Poisson brackets
- Interpret calculus of variations, Geodesics on a plane, Variational principle and Euler-Lagrange equation

6.2 PHASE SPACE AND LIOUVILLE'S THEOREM

In the Lagrangian formulation of mechanics, for describing the motion of a system having s degrees of freedom, the system at any instant of time t is represented by a point in an abstract s-dimensional mathematical space called the **configuration space** of the system. The point is called the **system point** at the instant t.

As time passes, the system point moves in the configuration space and it traces out, in general, a curve that gives the trajectory or path of the system.

In the Hamiltonian formulation, s generalized coordinates and s generalized momenta are independent variables for the system. An abstract s dimensional mathematical space, any point of which gives the s momenta of the system, is called the **momentum space** of the system. Clearly, any point in the momentum space represents the state of motion of the system at some instant of time. With progress of time, the point representing the state of motion moves in the momentum space. The curve traced out by the point is called **hodograph**.

To describe a function such as the Hamiltonian function $H(q, p, t)$ for the system we need a combination of the configuration space (coordinate space) and the momentum space for the system. Clearly, such a space is $2s$ dimensional. Such an abstract $2s$ dimensional mathematical space, any point of which represents the s coordinates (q_1, ….., qs) and s momenta (p_1, ….., ps) of the system at some instant of time, is called **phase space** of the system. Any point of the phase space describes not only the position of the system as a whole but also the state of motion of the system at some instant of time. As time passes, the point representing the configuration and the state of motion of the system in the phase space traces out a trajectory called the **phase trajectory**.

The concept of phase space and phase trajectory can be understood from the following example: consider a linear harmonic oscillator of mass m and oscillating along the X-axis.

The total energy of the oscillator when the displacement is *x* from the equilibrium position is given by

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2 x^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \qquad (1)$$

where $p = m\dot{x}$ is the momentum of the oscillator and $\omega = \sqrt{\frac{k}{m}}$ is the angular frequency of the oscillator.

We may express Equation (1) as

$$\frac{p^2}{2mE} + \frac{x^2}{\frac{2E}{m\omega^2}} = 1 \qquad (2)$$

In view of Equation (2), we find that the oscillator traverses an elliptical path in a two-dimensional space with *x* and *p* as the axes and having the semi-major axis equal to $\sqrt{\frac{2E}{m\omega^2}}$ and semi-minor axis equal to $\sqrt{2mE}$. In Figure 6.1 are shown different paths of the oscillator corresponding to different energies. We observe that the phase space of the oscillator is two-dimensional and phase trajectories are ellipses of different semi-axes corresponding to different total energies.

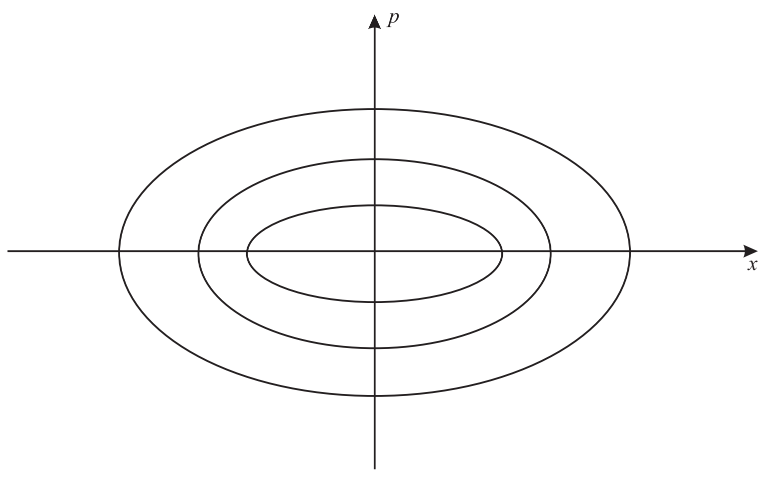

Fig. 6.1 *Paths of Oscillation*

NOTES

One important feature of phase trajectory is that no two phase trajectories can intersect with each other. This can be seen as follows: Let two trajectories cross at the phase point (xi,pi). If we consider this point to represent the position and the momentum at $t = 0$ then there will be two possible momenta along which the motion could start. This is not possible because the solutions of the oscillator equation

$$\ddot{x} + \omega^2 x = 0$$

at any instant for $\dot{x}$ and hence for p is unique.

6.2.1 Liouville's Theorem

The dynamical state of a mechanical system at any instant of time is represented by a point in the phase space of the system which is an imaginary mathematical space of 2s dimensions if 's' be the degrees of freedom of the system. As the system develops with time, the point representing the dynamical state called the representative point traces a path or trajectory determined by the Hamilton's canonical equations given by

$$\hat{q}_i = \frac{\partial H}{\partial p_i}, \ \hat{p}_i = \frac{-\partial H}{\partial p_i} \quad i = 1, \cdots, s$$

Where H is Hamiltonian of the system given as

$$H = (q_i, ..., q_s, p_i, ..., p_s, t)$$

As a result of motion, the density p (number of phase point per unit volume of the phase space at a given time in the phase space) changes with time. Our interest lies in the determination of the rate at which the density changes with time at a given point in the phase space. To obtain $\frac{dp}{dt}$ we use the theorem given by Liouville. The theorem consists of two parts:

(i) The density in phase space is a conserved quantity i.e., $\frac{dp}{dt} = 0$.

(ii) Extension in phase space is conserved i.e., $\frac{d}{dt}(\delta\Gamma) = 0$. This means that the volume available to a particular number of phase points is conserved throughout the phase space.

Proof of Theorem (1)

Consider the volume of phase space located between

q_1 and $q_1 + \delta q_1$, q_2 and $q_2 + \delta q_2$,..., q_s, and $q_s + \delta q_s$,

p_1 and $p_1 + \delta p_1$ ... , p_s and $p_s + \delta p_s$

The number of phase points located in this volume i.e., in the volume (δq, x ... x δq_s) $\times$(δp_1 ... x δp_s) changes as the coordinates q and momenta p vary with time.

NOTES

In a time dt, the change in the number of phase points in the above volume is equal to $\left(\frac{\partial p}{\partial t}\right) dt\ (\delta q_1\ x \ldots x\ \delta p_s)$.

The above change is due to the number of phase points entering and leaving the volume in the time dt.

Finding out the net increase δN in the number of phase in the above volume in time dt, we get the rate of increasing of δN to be given by

$$\frac{d}{dt}(\delta N) = -\sum_{i=1}^{s}\left\{\rho\left(\frac{\partial \dot{q}_i}{\partial q_i}+\frac{\partial \dot{p}_i}{\partial p_i}\right)\left(\frac{\partial p}{\partial q_i}\dot{q}_i+\frac{\partial p}{\partial p_i}\dot{p}_i\right)\right\}dt$$
$$\times\ \delta q_1\ x \ldots x\ \delta p_s$$

But $$\frac{d}{dt}(\delta N) = \frac{\partial p}{\partial t}dt\ \ \delta q_1 x \ldots \delta p_s$$

equating the above two relations we obtain

$$\frac{\partial p}{\partial t}dt\ \delta q_i\ x \cdots x\ \delta p_s = -\sum_{i=1}^{s}\left\{\rho\left(\frac{\partial \dot{q}_i}{\partial q_i}+\frac{\partial \dot{p}_i}{\partial p_i}\right)+\left(\frac{\partial p}{\partial q_i}\dot{q}_i\ .+\frac{\partial p}{\partial p_i}\dot{p}_i\right)\right\}\times dt$$
$$\times\ \delta q_1\ x \ldots x\ \delta p_s$$

or $$\frac{\partial p}{\partial t} = -\sum_{i=1}^{s}\left\{\rho\left(\frac{\partial \dot{q}_i}{\partial q_i}+\frac{\partial \dot{p}_i}{\partial p_i}\right)+\left(\frac{\partial p}{\partial q_i}\dot{q}_i.+\frac{\partial p}{\partial p_i}\dot{p}_i\right)\right\}$$

We have the Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i\ . = \frac{-\partial H}{\partial q_i}$$

Now $$\frac{\partial \dot{q}_i}{\partial q_i} = \frac{-\partial \dot{p}_i}{\partial p_i} = \frac{\partial^2 H}{\partial q_i\ \partial p_i}$$

Since the order of differential is immaterial (coordinates and momenta being independent variables we get

$$\sum_{i=1}^{s}\left(\frac{\partial p_i}{\partial q_i}+\frac{\partial \dot{p}_i}{\partial p_i}\right) = 0$$

Substituting the above in the expression for $\frac{\partial p}{\partial t}$ we get

$$\left(\frac{\partial p}{\partial t}\right)_{q,p} + \sum_{i=1}^{s}\frac{\partial p}{\partial q_i}\frac{dq_i}{dt} + \sum_{i=1}^{s}\frac{\partial p}{\partial p_i}\frac{dp_i}{dt} = 0$$

The above equation is identical with this equation of continuity in hydrodynamics. Considering $\rho = \rho(q, p, t)$ we get

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + \sum_{i=1}^{s} \frac{\partial p}{\partial q_i} \dot{q}_i . + \sum_{i=1}^{s} \frac{\partial p}{\partial p_i} \dot{p}_i$$

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Comparing the above two equations we get

$$\frac{dp}{dt} = 0$$

We thus find that the density in phase space is conserved.

Proof of Theorem (2)

We know that

$$\delta N = \rho \delta V$$

Taking total time derivative of the above equation we get

$$\frac{d}{dt}(\delta N) = \frac{dp}{dt} \delta V + \rho \frac{d}{dt}(\delta V)$$

Since the number of phase points in a given region of the phase space is invariant i.e., phase points are neither created nor destroyed, we get

$$\frac{d}{dt}(\delta N) = 0$$

Hence we get

$$\frac{dp}{dt} \delta V + \rho \frac{d}{dt}(\delta V) = 0$$

But $\frac{dp}{dt} = 0$ (Liouville's theorem (1)) and hence

$$\rho \frac{d}{dt}(\delta V) = 0$$

Since $\rho \neq 0$ we finally get

$$\frac{d}{dt}(\delta V) = 0$$

The above result represents conservation of extension in phase space.

Check Your Progress

1. Define configuration space.
2. What is momentum space?
3. What do you understand by hodograph?
4. What is phase trajectory?

NOTES

6.3 SPECIAL TRANSFORMATIONS

In Lagrangian formulation, a mechanical system of s degrees of freedom is described by a set of s independent generalized coordinates denoted as $(q_1, ..., q_s)$. The Lagrangian function of the system is, in general, a function of the generalized coordinates, the generalized velocities $\dot{q}_1, \dot{q}_2, ..., \dot{q}_s$ and time t. The generalized velocities are, however, not independent variables because they are the total time derivatives of the generalized coordinates. Thus, Lagrangian (L) of the system is a function of (s + 1) independent variables.

In Lagrangian formulation, there is no restriction to the choice of the generalized coordinates. In a given problem, these are so chosen that solution of the problem becomes mathematically simplified on the one hand and provides physical insight into the problem on the other (the choice is so made that most of the coordinates become cyclic or ignorable). The form of all the general relations (Lagrangian equations) remains the same for all sets of generalized coordinates. Any transformation from the set of coordinates $(q_1, ..., q_s)$ to the new set $(Q_1, ..., Q_s)$ represented by transformation equations

$$Q_k = Q_k(q_1, ..., q_s, t) \; ; \; (k = 1, ..., s) \tag{3}$$

leads to equations (Lagrangian equations) which are identical in form with those involving q's. In other words, Lagrange's equations are covariant to coordinate transformations which satisfy Equation (3) called point transformation.

In Hamiltonian formulation, on the other hand, the system is described by not only the s independent generalized coordinates but the description includes s independent generalized momenta $(p_1, ..., p_s)$ defined through the equations $p_k = \frac{\partial L}{\partial \dot{q}_k}$, $k = 1, ..., s$. The Hamiltonian function H of the system, in general, is a function of (2s + 1) independent variables, i.e.,

$$H = H(q_1, ... , q_s, p_1, ... , p_s, t)$$

and the Hamilton's canonical equations are

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \; ; \; \dot{p}_k = -\frac{\partial H}{\partial q_k}; \quad (k = 1, ..., s) \tag{4}$$

It is important to remember that the independent variables q's and p's are treated on equal footing or equal status, i.e., one is not considered to be more fundamental than the other in this formalism. Thus, for a reason similar to that used in the Lagrangian formulation, i.e., to work in the Hamiltonian formulation with advantage so as to gain mathematical simplicity in solving

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problems and to gain physical insight into the problems the concept of coordinate transformation should be generalized to include simultaneous transformation of the set of generalized coordinates $q_1, \ldots, q_s$ and the set of generalized momenta $p_1, \ldots, p_s$ to some new set $Q_1, \ldots, Q_s$ and $(P_1, \ldots, P_s)$, respectively related by transformation equations of the form

$$\begin{aligned} Q_k &= Q_k(q_1, \ldots, q_s, p_1, \ldots, p_s, t) \\ P_k &= P_k(q_1, \ldots, q_s, p_1, \ldots, p_s, t) \end{aligned} \tag{5}$$

These transformations are fundamentally different from those given by Equation (3), because here each new coordinate is a function of all the old coordinates, all the old momenta and time, and similarly each new momentum is a function of all the old coordinates, all the old momenta and time.

Thus, while point transformations refer to transformation of the configuration space of a system, the transformations of the type given by Equation (5) refer to transformation of the phase space of the system.

It is easily seen that in all possible transformations of the type given by Equation (5), the equations of motion are found not to retain their forms (canonical forms) as given in Equation (4).

In the development of Hamiltonian formalism, only those transformations find importance for which Q and P turn out to be canonical coordinates, i.e., for which the equations of motion of the form given by Equation (4) are satisfied

$$\dot{Q}_k = \frac{\partial K}{\partial P_k} \;;\; \dot{P}_k = -\frac{\partial K}{\partial Q_k};\; (k = 1, \ldots, s) \tag{6}$$

In the above, K a function of Q, P, and t in general, i.e.,

$$K = K(Q_1, \ldots, Q_s, P_1, \ldots, P_s, t) = K(Q, P, t),$$

which plays the role of the Hamiltonian function in the transformed set of coordinates and momenta.

The function K is defined according to

$$K = K\;Q, P, t\; = \sum_k P_k \dot{Q}_k - L'\left(Q, \dot{Q}, t\right) \tag{7}$$

where $L'\left(Q, \dot{Q}, t\right)$ is a function which when substituted in Hamilton's principle

$$\delta \int_1^2 L'\, dt = 0 \tag{8}$$

gives the correct equations of motion in terms of the new set of coordinates $\{Qk\}$, namely,

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{Q}_k}\right) = \frac{\partial L'}{\partial Q_k} \;;\; \left(k = 1, ..., s\right) \tag{9}$$

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All such transformations which conform to the above conditions are referred to as contact, or canonical, transformations.

We may note the following:

(i) In point transformation, the new Lagrangian can be obtained from the old by direct substitution of the transformation relations. Such direct relationship between the new and old Lagrangian functions may not exist in canonical transformation.

(ii) Point transformations are made keeping in view the particular mechanical problem under consideration. Canonical transformations are, however, problem independent. For all mechanical systems having the same number of degrees of freedom, the numbers of transformed canonical coordinates (Q, P) are the same.

6.3.1 Generating Function for Canonical Transformation

The Hamiltonian in the old sets of coordinates and momenta is

$$H = \sum p_k \dot{q}_k - L\left(q, \dot{q}, t\right) \tag{10}$$

The principle of least action, $\delta \int_{t_1}^{t_2} L dt = 0$, can alternatively be written in the modified form (using Equation (10)) as

$$\delta \int_{t_1}^{t_2} \left[\sum p_k \dot{q}_k - H \right] dt = 0$$

or

$$\delta \int_{t_1}^{t_2} \left[\sum p_k dq_k - H dt \right] = 0 \tag{11}$$

For the transformed coordinates $\{Qk\}$ and momenta $\{Pk\}$ to be canonical they must also satisfy the modified Hamilton's principle

$$\delta \int_{t_1}^{t_2} \left[\sum P_k dQ_k - K dt \right] = 0 \tag{12}$$

The right hand sides of both the Equation (11) and (12) are zero. This fact, however, does not mean that the integrands of the integrals in the Equation (11) and (12) are equal. Subtracting Equation (12) from Equation (11) we obtain

$$\delta \int_{t_1}^{t_2} \left[\sum p_k dq_k - H dt - \sum P_k dQ_k - K dt \right] = 0$$

or $$\delta\int_{t_1}^{t_2}\left[\left(\sum p_k\,\dot{q}_k - H\right)-\left(\sum P_k\,\dot{Q}_k - K\right)\right]dt = 0 \tag{13}$$

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Let us now consider a function F of coordinates, momenta and time, i.e., $F = F\ q,p,t$. Hence, we have

$$\delta\int_{t_1}^{t_2}\frac{dF\ q,p,t}{dt}dt = \delta\ F\ q,p,t\ \Big|_{t_1}^{t_2}$$

$$\sum\left\{\frac{\partial F}{\partial q_k}\delta q_k\right\}_{t_1}^{t_2} + \sum\left\{\frac{\partial F}{\partial p_k}\delta p_k\right\}_{t_1}^{t_2} + \left\{\frac{\partial F}{\partial t}\delta t\right\}_{t_1}^{t_2} = 0 \tag{14}$$

since at the end points t_1 and t_2, we have $\delta q_k = 0$, $\delta p_k = 0$ and $\delta t = 0$.

In view of the above, we find that Equation (13) is not affected if we add $\delta\int_{t_1}^{t_2}\frac{dF}{dt}dt$ to it or subtract from it.

We hence get

$$\delta\int_{t_1}^{t_2}\left[\left(\sum p_k\,\dot{q}_k - H\right)-\left(\sum P_k\,\dot{Q}_k - K\right)\right]dt - \delta\int_{t_1}^{t_2}\frac{dF}{dt}dt = 0$$

or $$\delta\int_{t_1}^{t_2}\left[\left(\sum p_k\,\dot{q}_k - H\right)-\left(\sum P_k\,\dot{Q}_k - K\right)\right]dt = \delta\int_{t_1}^{t_2}\frac{dF}{dt}dt$$

We may now equate the integrands of the integrals on the two sides of the above equation to get

$$\sum p_k\,\dot{q}_k - H - \sum P_k\,\dot{Q}_k + K = \frac{dF}{dt}$$

or $$\sum_k p_k\,\dot{q}_k - \sum_k P_k\,\dot{Q}_k + K - H = \frac{dF}{dt} \tag{15}$$

From Equation (15) it follows that F is, in general, a function of ($4s$ + 1) variables which are $2s$ old coordinates and momenta, $2s$ new coordinates and momenta and time. However, there exist $2s$ transformation relations given by Equation (5), and hence F is effectively a function of $2s + 1$ independent variables.

These variables are: time t and any $2s$ variables out of the total $4s$. The $2s$ variables may be chosen as (*i*) s old coordinates and s new coordinates, (*ii*) s old coordinates and s new momenta, (*iii*) s old momenta and s new coordinates, and (*iv*) s old momenta and s new momenta and as such, the function F may have the following forms:

(*i*) $F_1(q, Q, t)$, (*ii*) $F_2(q, P, t)$, (*iii*) $F_3(p, Q, t)$, (*iv*) $F_4(p, P, t)$.

F being a function of old and new variables affects transformation from old sets of coordinates and momenta to new sets of coordinates and momenta. In other words, the transformation relations given by Equation (5) can be derived from a knowledge of the function F. For this reason, the function F is termed the **generating function** of the canonical transformation under consideration.

6.3.2 Transformation Relations for Different Forms of the Generating Function

In the following we will derive the transformation relations between the old sets of coordinates and momenta with new sets of coordinates and momenta when canonical transformation is affected by the generating functions having the four forms as mentioned in the last section.

(*a*) Generating Function of the Form

$$F_1 = F_1\; q, Q, t$$

From Equation (15) we have

$$\sum p_k \dot{q}_k - \sum P_k \dot{Q}_k + K - H = \frac{dF_1}{dt} \qquad (16)$$

Since $F_1 = F_1\; q, Q, t\; = F_1\; q_1, ..., q_s, Q_1, ... Q_s, t$, we get

$$\frac{dF_1}{dt} = \sum_k \frac{\partial F_1}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial F_1}{\partial Q_k} \dot{Q}_k + \frac{\partial F_1}{\partial t} \qquad (17)$$

Using Equation (17) in Equation (16) we obtain

$$\sum p_k \dot{q}_k - \sum P_k \dot{Q}_k + K - H = \sum_k \frac{\partial F_1}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial F_1}{\partial Q_k} \dot{Q}_k + \frac{\partial F_1}{\partial t}$$

Equating the coefficients of $\dot{q}_k$ and $\dot{Q}_k$ on both sides of the above equation we obtain

$$p_k = \frac{\partial F_1}{\partial q_k} \qquad (18)$$

or

$$P_k = -\frac{\partial F_1}{\partial Q_k} \qquad (19)$$

We also obtain

$$K - H = \frac{\partial F_1}{\partial t} \quad \text{or} \quad K = H + \frac{\partial F_1}{\partial t} \qquad (20)$$

(*b*) *Generating Function of the Form*

$F_2(q, P, t)$ can be obtained from $F_1(q, Q, t)$ by means of a change of the basis of description from (q, Q) to (q, P). This can be done using Legendre transformation discussed in Section 4.2. The result is

$$F_2(q, P, t) = F_1(q, Q, t) + \sum P_k Q_k \qquad (21)$$

Using Equation (19) in Equation (21), we get

$$\frac{\partial F_2}{\partial Q_k} = \frac{\partial F_1}{\partial Q_k} + P_k = -P_k + P_k = 0$$

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Thus, the function F_2 is indeed independent of the new coordinates. From Equation (21) we get

$$F_1(q, Q, t) = F_2(q, P, t) - \sum_k P_k Q_k \tag{22}$$

Using Equation (22) in Equation (15), we obtain

$$\sum_k p_k \dot{q}_k - \sum_k P_k \dot{Q}_k + (K - H) = \frac{dF_1}{dt} = \frac{d}{dt}\left[F_2(q, P, t) - \sum_k P_k Q_k\right]$$

or
$$\sum_k p_k \dot{q}_k - \sum_k P_k \dot{Q}_k + (K - H)$$

$$= \sum_k \frac{\partial F_2}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial F_2}{\partial P_k}\dot{P}_k - \sum_k \dot{P}_k Q_k + \frac{\partial F_2}{\partial t} - \sum_k P_k \dot{Q}_k$$

or
$$\sum_k p_k \dot{q}_k + (K - H) = \sum_k \frac{\partial F_2}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial F_2}{\partial P_k}\dot{P}_k - \sum_k \dot{P}_k Q_k + \frac{\partial F_2}{\partial t}$$

Comparing the coefficients of $\dot{q}_k$ and $\dot{P}_k$ on the two sides of the above equation, we get

$$p_k = \frac{\partial F_2}{\partial q_k} \tag{23}$$

$$Q_k = \frac{\partial F_2}{\partial P_k} \tag{24}$$

We also obtain

$$K - H = \frac{\partial F_2}{\partial t} \text{ or } K = H + \frac{\partial F_2}{\partial t} \tag{25}$$

(*c*) *Generating Function of the Form*

As in the previous case, $F_3(p, Q, t)$ is obtained from $F_1(q, Q, t)$ by a change of the basis of description from (q, Q) to (p, Q) which is affected using Legendre transformation. We get

$$F_3(p, Q, t) = F_1(q, Q, t) - \sum_k p_k q_k \tag{26}$$

We observe

$$\frac{\partial F_3}{\partial q_k} = \frac{\partial F_1}{\partial q_k} - p_k$$

Using Equation (18), the above gives

$$\frac{\partial F_3}{\partial q_k} = p_k - p_k = 0 \tag{27}$$

We observe that F_3 is indeed independent of old coordinates. Substitution of Equation (26) in Equation (15) gives

$$\sum_k p_k \dot{q}_k - \sum_k P_k \dot{Q}_k + (K - H) = \frac{d}{dt}\left[F_3(p, Q, t) + \sum_k p_k q_k\right]$$

$$= \sum_k \frac{\partial F_3}{\partial p_k}\dot{p}_k + \sum_k \frac{\partial F_3}{\partial Q_k}\dot{Q}_k + \frac{\partial F_3}{\partial t} + \sum_k p_k \dot{q}_k + \sum_k \dot{p}_k q_k$$

Simplifying, we obtain

$$-\sum P_k \dot{Q}_k + (K - H) = \sum \frac{\partial F_3}{\partial p_k}\dot{p}_k + \sum \frac{\partial F_3}{\partial Q_k}\dot{Q}_k + \sum \dot{p}_k q_k + \frac{\partial F_3}{\partial t} \tag{28}$$

Equating coefficients of $\dot{Q}_k$ and $\dot{p}_k$ on both sides of the above equation we get

$$-P_k = \frac{\partial F_3}{\partial Q_k} \text{ or } P_k = -\frac{\partial F_3}{\partial Q_k} \tag{29}$$

$$q_k + \frac{\partial F_3}{\partial p_k} = 0 \text{ or } q_k = -\frac{\partial F_3}{\partial p_k} \tag{20}$$

We also obtain

$$K - H = \frac{\partial F_3}{\partial t} \text{ or } K = H + \frac{\partial F_3}{\partial t} \tag{31}$$

(*d*) *Generating Function of the Form*

$F_4(p, P, t)$ can be obtained from $F_1(q, Q, t)$ by a change of the basis of description from (q, Q) to (p, P) which can be affected using double Legendre transformation. The result is

$$F_4(p, P, t) = F_1(p, Q, t) + \sum_k P_k Q_k - \sum_k p_k q_k \tag{32}$$

The above gives on using Equation (18)

$$\frac{\partial F_4}{\partial q_k} = \frac{\partial F_1}{\partial q_k} - p_k = p_k - p_k = 0$$

$$\frac{\partial F_4}{\partial Q_k} = \frac{\partial F_1}{\partial Q_k} + P_k = -P_k + P_k = 0 \qquad \text{[using Equation (19)]}$$

Clearly, F_4 is not a function of old as well as new coordinates.

Substitution of Equation (32) in Equation (15) gives

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$$\sum_k p_k\,\dot{q}_k - \sum_k P_k\,\dot{Q}_k + K - H = \frac{d}{dt}\left[F_4\left(p, P, t\right) - \sum_k P_k Q_k + \sum_k p_k\, q_k\right]$$

$$= \sum_k \frac{\partial F_4}{\partial p_k}\,\dot{p}_k + \sum_k \frac{\partial F_4}{\partial P_k}\,\dot{P}_k + \frac{\partial F_4}{\partial t} - \sum_k P_k\,\dot{Q}_k - \sum_k \dot{P}_k\, Q_k + \sum_k p_k\,\dot{q}_k + \sum_k \dot{p}_k\, q_k$$

Simplifying, we get from the above

$$K - H = \sum_k \frac{\partial F_4}{\partial p_k}\,\dot{p}_k + \sum_k \frac{\partial F_4}{\partial P_k}\,\dot{P}_k - \sum_k \dot{P}_k\, Q_k + \sum_k \dot{P}_k\, q_k + \frac{\partial F_4}{\partial t} \tag{33}$$

Equating coefficients of $\dot{p}_k$ and $\dot{P}_k$ on both sides of the above equation we obtain

$$0 = \frac{\partial F_4}{\partial p_k} + q_k \quad \text{or} \quad q_k = -\frac{\partial F_4}{\partial p_k} \tag{34}$$

$$0 = \frac{\partial F_4}{\partial P_k} - Q_k \quad \text{or} \quad Q_k = \frac{\partial F_4}{\partial P_k} \tag{35}$$

Equation (34) also gives

$$K - H = \frac{\partial F_4}{\partial t} \quad \text{or} \quad K = H + \frac{\partial F_4}{\partial t} \tag{36}$$

From the results obtained above we may note that irrespective of the form of the generating function F affecting the canonical transformation

$$K = H + \frac{\partial F}{\partial t}$$

If F is not an explicit function of time, then $\frac{\partial F}{\partial t} = 0$ so that $K = H$. Clearly, the transformed Hamiltonian K is obtained by substituting the values of p and q in terms of the transformed variables P and Q in the expression for H.

6.3.3 Examples of Physically Meaningful Canonical Transformations

1. Consider a Canonical Transformation Affected by the Generating Function

$$F = \sum q_k\; P_k \tag{37}$$

The function F is of the form $F_2\left(q, P, t\right)$. Hence, Equation (23), (24) and (25) hold and we have

$$p_k = \frac{\partial F}{\partial q_k}\;,\; Q_k = \frac{\partial F}{\partial p_k} \;\text{ and }\; K = H + \frac{\partial F}{\partial t} \tag{38}$$

Since $F = \sum q_k P_k$, we have the relations

$$\frac{\partial F}{\partial q_k} = P_k\;,\; \frac{\partial F}{\partial P_k} = q_k \;\text{ and }\; K = H \tag{39}$$

The above two sets of relations give

$$Q_k = q_k \ , \ P_k = p_k \ \text{ and } \ K = H \tag{40}$$

so that the above generating function simply generates an identity transformation.

2. Consider a Canonical Transformation affected by the Generating Function

$$F = -\sum_k q_k P_k \tag{41}$$

The above generating function is also of the form $F_2(q, P, t)$. Hence, the relations given by Equation (23), (24) and (25) hold and we have the relations

$$p_k = \frac{\partial F}{\partial q_k} \ , \ Q_k = \frac{\partial F}{\partial P_k} \ \text{ and } \ K = H + \frac{\partial F}{\partial t} \tag{42}$$

From $F = -\sum q_k P_k$, we obtain

$$\frac{\partial F}{\partial q_k} = -P_k \ , \ \frac{\partial F}{\partial P_k} = -q_k \ \text{ and } \ \frac{\partial F}{\partial t} = 0 \tag{43}$$

The above two sets of relations yield

$$Q_k = -q_k \ , \ P_k = -p_k \ \text{ and } \ K = H \tag{44}$$

Clearly, the above generating function generates transformation leading to space inversion.

3. Consider the Canonical Transformation Affected by the Generating Function

$$F = \sum_k q_k Q_k$$

The given generating function is of the form $F_1(q, Q, t)$. Hence, the relations given by Equation (18), (19) and (20) hold and we have the relations

$$p_k = \frac{\partial F}{\partial q_k} \ , \ P_k = -\frac{\partial F}{\partial Q_k} \ \text{ and } \ K = H + \frac{\partial F}{\partial t} \tag{45}$$

Since $F = \sum q_k Q_k$, we get

$$\frac{\partial F}{\partial q_k} = Q_k \ , \ \frac{\partial F}{\partial Q_k} = q_k \ \text{ and } \ \frac{\partial F}{\partial t} = 0 \tag{46}$$

From the above two sets of relations, we obtain

$$p_k = Q_k \ , \ P_k = -q_k \ \text{ and } \ K = H \tag{47}$$

Clearly, the above generating function interchanges the momenta and coordinates. This illustrates the independent status or footing of generalized coordinates and generalized momenta.

4. Consider a Generating Function of Canonical Transformation of the Type

$$F = \sum_k f_k\left(q_1, \ldots.. q_s, t\right) P_k \tag{48}$$

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The new transformed coordinates are given by

$$Q_k = \frac{\partial F}{\partial P_k} = f_k\left(q_1, \ldots.. q_s, t\right) \tag{49}$$

We find that the new coordinates are functions of the old coordinates. In other words, the transformation generated by the function F of the type given above corresponds to a point transformation. Since the function $f_k\left(q_1, \ldots.. q_s, t\right)$ is arbitrary we can remark that all point transformations are canonical.

6.3.4 Conditions for Canonical Transformation

(*i*) Consider the canonical transformation from old set of coordinates and momenta $\{q, p\}$ to a new set of coordinates and momenta $\{Q, P\}$, i.e., consider the canonical transformations.

$$\begin{aligned} Q_k &= Q_k\left(q_1, \ldots.. q_s, p_1, \ldots\ldots p_s, t\right) \\ P_k &= P_k\left(q_1, \ldots.. q_s, p_1, \ldots.. p_s, t\right) \end{aligned} \quad ; \quad k = 1, 2, \ldots.., s \tag{50}$$

Let the generating function which affects the above transformation when time is held fixed be of the form $F = F\left(q, Q\right)$. For this form of generating function it has been shown that

$$p_k = \frac{\partial F}{\partial q_k} \tag{51}$$

and

$$P_k = -\frac{\partial F}{\partial Q_k} \tag{52}$$

Further, since $F = F\left(q, Q\right)$ we get the total differential of F to be

$$dF = \sum_k \frac{\partial F}{\partial q_k} dq_k + \sum_k \frac{\partial F}{\partial Q_k} dQ_k \tag{53}$$

Using Equation (51) and (52) in the above, we get

$$dF = \sum p_k dq_k - \sum P_k dQ_k \tag{54}$$

Since dF is an exact differential, the expression on the r.h.s. of the above equation

$$\sum p_k dq_k - \sum P_k dQ_k \tag{55}$$

must also be an exact differential. The above condition of exact differential of the expression $\sum p_k dq_k - \sum P_k dQ_k$ can also be obtained for any other form of generating function affecting the canonical transformation from $\{q, p\}$ to $\{Q, P\}$, or vice-versa.

(*ii*) For a transformation from $\{q, p\}$ to $\{Q, P\}$ to be canonical, the bilinear form $\sum_k \delta p_k dq_k - \delta q_k dp_k$ must remain invariant, i.e., we must have

$$\sum_k \delta p_k dq_k - \delta q_k dp_k = \sum_k \delta P_k dQ_k - \delta Q_k dP_k$$

Proof: We have the Hamilton's canonical equations

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \quad (56)$$

and $$\dot{p}_k = -\frac{\partial H}{\partial q_k} \quad (57)$$

The above equations give

$$dq_k = \frac{\partial H}{\partial p_k} dt \quad (58)$$

and $$dp_k = -\frac{\partial H}{\partial q_k} dt \quad (59)$$

From Equation (58) and (59) we get

$$\delta p_k \left(dq_k - \frac{\partial H}{\partial p_k} dt \right) = 0 \quad (60)$$

and $$\delta q_k \left(dp_k - \frac{\partial H}{\partial q_k} \right) dt = 0 \quad (61)$$

where δpk and δqk are arbitrary variations of pk and qk, respectively.

Summing over all the degrees of freedom, the above equations yield

$$\sum_k \left[\delta p_k \left(dq_k - \frac{\partial H}{\partial p_k} dt \right) \right] - \sum_k \left[\delta q_k \left(dp_k - \frac{\partial H}{\partial q_k} dt \right) \right] = 0$$

or $$\sum_k \delta p_k dq_k - \delta q_k dp_k - \sum_k \left(\frac{\partial H}{\partial p_k} \delta p_k + \frac{\partial H}{\partial q_k} \delta q_k \right) dt = 0$$

or $$\sum_k \delta p_k dq_k - \delta q_k dp_k - \delta H dt = 0 \quad (62)$$

Similarly, we obtain using the transformed coordinates and momenta

$$\sum_k \delta P_k dQ_k - \delta Q_k dP_k - \delta K dt = 0 \quad (63)$$

If we assume the generating function affecting the canonical transformation not to be an explicit function of time, we get $K = H$. We then get from Equation (62) and (63)

$$\sum_k \delta p_k dq_k - \delta q_k dp_k = \sum_k \delta P_k dQ_k - \delta Q_k dP_k \quad (64)$$

Clearly, for canonical transformation, the bilinear form remains invariant.

6.3.5 Infinitesimal Transformation

NOTES

The concept of **infinitesimal contact**, **or canonical transformation**, has been found to be immensely useful in mechanics. The motion of a mechanical system during a finite interval of time can be looked upon as a succession of infinitesimal transformations of the motion generated by the Hamiltonian of the system. This can be understood from the following:

Consider a mechanical system of s degrees of freedom described by generalized coordinates $q_1, ..., q_s$ and generalized momenta $p_1, ..., p_s$. Let us affect a canonical transformation of the coordinates and momenta to new sets $Q_1, ..., Q_s$ and $P_1, ..., P_s$, respectively.

The canonical transformation is said to be an infinitesimal transformation if the new coordinates differ from the old coordinates by infinitesimal amounts and similarly the new momenta differ from the old momenta by infinitesimal amounts. Clearly, corresponding to infinitesimal transformations, the transformation equations are of the form

$$\begin{aligned} Q_k &= q_k + \delta q_k \\ P_k &= p_k + \delta p_k \end{aligned} \; ; \quad k = 1, 2, ..., s \tag{65}$$

where δqk and δpk are respectively the infinitesimal changes in the coordinate qk and momentum pk.

We know that canonical transformation affected by the generating function $\sum q_k P_k$ is an identity transformation leading to $Qk = qk$ and $Pk = pk$ (**see Section 6.3.3**). This result allows us to consider the generating function affecting in infinitesimal transformation to differ from the generating function $\sum q_k P_k$ by an infinitesimal amount. As such, the generating function for infinitesimal transformation can be expressed as

$$F = \sum q_k \; P_k + \varepsilon G \;\; q, p, t \tag{66}$$

In the above, ε is an infinitesimal parameter related to the transformation and G (q, p, t) is any differentiable function of $(2s + 1)$ variables.

F being of the from $F_2 (q, p, t)$, we have according to Equation (23)

$$p_k = \frac{\partial F}{\partial q_k} \tag{67}$$

From Equation (66) we get

$$\frac{\partial F}{\partial q_k} = P_k + \varepsilon \frac{\partial G}{\partial q_k} \tag{68}$$

Combining Equation (67) and (68), we obtain

$$p_k = P_k + \varepsilon \frac{\partial G}{\partial q_k} \tag{69}$$

NOTES

Now,

$$\delta p_k = P_k - p_k$$

Hence, using Equation (69), we get

$$\delta p_k = -\varepsilon \frac{\partial G}{\partial q_k} \qquad (70)$$

Again, according to Equation (67), we have

$$Q_k = \frac{\partial F}{\partial p_k} = q_k + \varepsilon \frac{\partial G}{\partial p_k} \qquad \text{[using Equation (66)]} \qquad (71)$$

The second term in the above equation is linear in ε. Besides, *Pk* differs from *pk* by an infinitesimal amount. Hence, without introducing any serious error we may replace $\varepsilon \frac{\partial G}{\partial P_k}$ by $\varepsilon \frac{\partial G}{\partial p_k}$. We may note that the difference involved in such replacement is only of the second order of smallness in the infinitesimal transformation parameter ε. We can hence write Equation (71) as

$$Q_k = q_k + \varepsilon \frac{\partial G}{\partial p_k}$$

so that we obtain

$$\delta q_k = Q_k - q_k = \varepsilon \frac{\partial G}{\partial p_k} \qquad (72)$$

From the knowledge of δ*qk* and δ*pk* as can be calculated using Equation (70) and (72) for each *k*, the values of the transformed coordinates and momenta can be calculated.

An important case of infinitesimal transformation arises when the generating function $G(q, p, t)$ affecting the transformation is the Hamiltonian $H(q, p, t)$ of the system and the parameter ε is an infinitesimal interval dt of time t. We then get from Equation (72) and Equation (70)

(*i*) $\delta q_k = dt \frac{\partial H}{\partial p_k}$

Using Hamilton's equation $\dot{q}_k = \frac{\partial H}{\partial p_k}$, the above becomes

$$\delta q_k = dt\, \dot{q}_k = dt \frac{dq_k}{dt} = dq_k \qquad (73)$$

(*ii*) $\delta p_k = -dt \frac{\partial H}{\partial q_k} = -dt\left(-\dot{p}_k\right)$

or

$$\delta p_k = dt \frac{dp_k}{dt} = dp_k \qquad (74)$$

Equations (73) and (74) tell us that the infinitesimal transformation causes the coordinates and momenta at the instant t to change to their respective values at the time $t + dt$. In other words, infinitesimal transformation generated by the Hamiltonian function describes the motion of the system during the interval dt. Hence, a succession of such infinitesimal transformations describes the motion in any finite time.

NOTES

6.3.6 More about Infinitesimal Transformation

In the following we will show that constants of motion of a mechanical system are the generating functions that do not alter the Hamiltonian of the system.

Consider a function $f(q, p)$ of generalized coordinates q and generalized momenta p.

Let the system point in the phase space of the system undergo an infinitesimal change so that the numerical values of the variables q and p also undergo some change. As an example, let us consider the changes in the variables to be a result of infinitesimal canonical transformations generated by the function $G(q, p)$. The change in the function f is given by

$$\delta f = f\left(q+\delta q, p+\delta p\right) - f\left(q, p\right)$$

The above can also be expressed as

$$\delta f = \sum_k \frac{\partial f}{\partial q_k}\delta q_k + \sum \frac{\partial f}{\partial p_k}\delta p_k \tag{75}$$

Using Equation (70) and (72) in the above, we obtain

$$\delta f = \sum \frac{\partial f}{\partial q_k}\varepsilon\frac{\partial G}{\partial p_k} + \sum \frac{\partial f}{\partial p_k}(-\varepsilon)\frac{\partial G}{\partial q_k}$$

$$\delta f = \varepsilon\left[\left(\frac{\partial f}{\partial q_k}\frac{\partial G}{\partial p_k}\right) - \right]\left(\frac{\partial f}{\partial p_k}\frac{\partial G}{\partial q_k}\right)$$

or

$$\delta f = \varepsilon\left[f, G\right] \tag{76}$$

where $[f, G]$ is the Poisson bracket of the functions f and G. Replacing the function f by the Hamiltonian H of the system in Equation (76), we get

$$\delta H = \varepsilon\left[H, G\right] \tag{77}$$

If the function G is a constant of motion, we have $[H, G] = 0$, and hence $\delta H = 0$. We thus find that a constant of motion generates an infinitesimal canonical transformation under which the Hamiltonian remains invariant. Alternatively, we can say that any infinitesimal canonical transformation which keeps the Hamiltonian invariant is generated by a function which is a constant of motion.

Check Your Progress

5. Write the different forms of the generating function of the canonical transformation under consideration.
6. What is the generating function of the form F_1 (q, Q, t)?
7. Write the generating function of the form F_2 (q, p, t).
8. What is the generating function of the form F_3 (p, Q, t).
9. Write the generating function of the form F_4 (p, p, t).
10. Define infinitesimal transformation.
11. Write the form of the transformation equations corresponding to infinitesimal transformations.

NOTES

6.4 LAGRANGE BRACKETS

Consider a mechanical system of s degree of freedom. Let the system be described by generalized coordinates q_1, ….., qs and conjugate momenta p_1, ….., ps.

Let $f = f(q_1, ..., qs, p_1, ..., ps) = f(q, p)$ and $g = g(q_1, ..., qs, p_1, ..., ps) = g(q, p)$ be two dynamical variables of the system.

The Lagrange bracket of f and g with respect to the basis (q, p) is written as $\{f, g\}q_{,} p$ and is defined as

$$\{f, g\}q_{,} p = \sum_k \left(\frac{\partial q_k}{\partial f}\frac{\partial p_k}{\partial g} - \frac{\partial p_k}{\partial f}\frac{\partial q_k}{\partial g} \right) = 0 \qquad (78)$$

We may note the following

(*a*) Taking $f = qi$ and $g = qj$ Equation (78) gives

$$\{qi, qj\} = \sum_k \left(\frac{\partial q_k}{\partial q_i}\frac{\partial p_k}{\partial q_j} - \frac{\partial p_k}{\partial q_i}\frac{\partial q_k}{\partial q_j} \right) = 0 \qquad (79)$$

(*b*) Taking $f = pk$ and $g = pj$ Equation (78) gives

$$\{pk, pj\} = \sum_k \left(\frac{\partial q_k}{\partial p_k}\frac{\partial p_k}{\partial p_j} - \frac{\partial p_k}{\partial p_k}\frac{\partial q_k}{\partial p_j} \right) = 0 \qquad (80)$$

(*c*) Taking $f = qk$ and $g = pj$ Equation (78) gives

$$\{qk, pj\} = \sum_k \left(\frac{\partial q_k}{\partial q_k}\frac{\partial p_k}{\partial p_j} - \frac{\partial p_k}{\partial q_k}\frac{\partial q_k}{\partial p_j} \right) = \delta_{kj} \qquad (81)$$

We may further note the following important **properties of Lagrange bracket**.

NOTES

(*d*) Lagrange bracket is invariant under canonical transformation from the set of variables (q, p) to the set of variables (Q, P), i.e.,

$$\{f, g\}_{q,p} = \{f, g\}_{Q,P} \tag{82}$$

(*e*) Lagrange bracket is non-commutative. Thus, we have

$$\{f, g\} = -\{g, f\} \tag{83}$$

(*f*) The following general theorem that relates Lagrange bracket and Poisson bracket is found to hold

$$\sum_{k=1}^{2n}\{f_k f_i\}\left[f_k, f_j\right] = \delta ij \tag{84}$$

where $f, f_2, ..., f_2n$ is a set of $2n$ independent functions, each of which is itself a function of n coordinates $q_1, ….., qn$ and n momenta $p_1, ….., pn$.

6.5 POISSON BRACKET

Consider a mechanical system of s degrees of freedom. Let $q_1, ..., qs$ be the generalized coordinates, and $p_1, ..., ps$ be the generalized momenta in terms of which the system is described. Let F be any dynamical variable of the system which is a function of the coordinates, momenta and time, i.e.,

$$F = F(q_1, ….., qs, p_1, ….., ps, t) = F(q, p, t) \tag{85}$$

The total time derivative of F is given by

$$\frac{dF}{dt} = \sum_k \frac{\partial F}{\partial q_k}\dot{q}_k + \sum \frac{\partial F}{\partial p_k}\dot{p}_k + \frac{\partial F}{\partial t} \tag{86}$$

Using the Hamilton's canonical equations given by

$$\dot{q}_k = \frac{\partial H}{\partial p_k};\ \dot{p}_k = \frac{-\partial H}{\partial q_k}$$

in Equation (86), we obtain

$$\frac{dF}{dt} = \sum_k \frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k} - \sum \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial q_k} + \frac{\partial F}{\partial t}$$

or

$$\frac{dF}{dt} = \sum_k \left[\frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial q_k}\right] + \frac{\partial F}{\partial t}$$

or

$$\frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t} \tag{87}$$

The quantity within the parenthesis on the right hand side of Equation (87) turns out to be of fundamental importance in the formal development of mechanics and is known as the Poisson bracket (PB) of F and H. It is usual to write it as $[F, H]_{q,p}$. Thus,

$$[F, H]_{q,p} = \sum_k \left[\frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial q_k}\right] \tag{88}$$

In general, for any two arbitrary physical quantities *f* and *g*, which are functions of coordinates, momenta and time, the Poisson bracket is defined as

$$[f, g]_{q,p} = \sum_k \left[\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k}\right] \tag{89}$$

Some of the special cases of Equation (87) give

(*i*) $\dot{q}_k = [qk, H]$ (choosing $F = qk$) (90)

(*ii*) $\dot{p}_k = p_k, H$ (choosing $F = p_k$) (91)

Again, from Equation (89) it follows that the PB of a quantity with itself is zero. Hence, we obtain from Equation (87)

(*iii*) $\dot{H} = \frac{dH}{dt} = \frac{\partial H}{\partial t} + H, H = \frac{\partial H}{\partial t}$, (choosing $F = H$) (92)

We may note the following identities from the general definition of Poisson bracket given by Equation (89).

$$\begin{aligned} [f, g] &= -[g, f] \\ [f, c] &= 0 \\ [cf, g] &= c[f, g] \end{aligned} \tag{93}$$

In the above, c is a constant

$$[f, g_1 + g_2] = [f, g_1] + [f, g_2]$$

$$[f, g_1g_2] = g_1[f, g_2] + [f, g_1]g_2 \tag{94}$$

$$\frac{\partial}{\partial t} f, g = \left[f, \frac{\partial g}{\partial t}\right] + \left[\frac{\partial f}{\partial t}, g\right] \tag{95}$$

Furthermore, some of the special cases of Equation (92) and (93) are easily seen to follow.

(*a*) Taking $g = qj$ in Equation (89), we get

$$[f, qj] = \sum_k \left[\frac{\partial f}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial q_j}{\partial q_k}\right]$$

$$= -\sum_k \frac{\partial f}{\partial p_k}\delta_{jk} \qquad \left[\because \frac{\partial q_j}{\partial p_k} = 0 \ ; \ \begin{aligned} \delta_{jk} &= 0 \text{ if } j \neq k \\ &= 1 \text{ if } j = k \end{aligned}\right]$$

Thus, $$[f, qj] = -\frac{\partial f}{\partial p_j} \tag{96}$$

(*b*) Taking $f = qk$ and $g = qj$, we obtain from Equation (89)

$$[qk, qj] = -\frac{\partial q_k}{\partial p_j} = 0 \tag{97}$$

Similarly, taking $g = qj$, we obtain

$$[pk, qj] = -\frac{\partial p_k}{\partial p_j} = -\delta_{kj} \tag{98}$$

(*c*) If $g = pj$, we get from Equation (93)

$$[f, pj] = \sum_k \left[\frac{\partial f}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial p_j}{\partial q_k}\right] = \sum \frac{\partial f}{\partial q_k}\delta_{jk}$$

or
$$[f, pj] = \frac{\partial f}{\partial q_j} \tag{99}$$

If $g = pj$, then

$$[pi, pj] = \frac{\partial p_i}{\partial p_j} = 0 \tag{100}$$

If $f = qi$, then

$$[qi, pj] = \frac{\partial q_i}{\partial q_j} = \delta_{ij} \tag{101}$$

The above results are summarized as

$$[qi, qj] = 0 = [pi, pj] \tag{102}$$

$$[qi, pj] = \delta ij$$

Equations (102) are known as the **Fundamental, or basic, Poisson brackets**.

6.5.1 Constants or Integrals of Motion

Consider Equation (87) which gives the total time derivative of the dynamical variable $F = F(q, p, t)$

$$\frac{dF}{dt} = \sum_k \left(\frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial q_k}\right) + \frac{\partial F}{\partial t}$$

or
$$\frac{dF}{dt} = [F, H]_{q,p} + \frac{\partial F}{\partial t} \tag{103}$$

If F has no explicit time dependence, we have $\frac{\partial F}{\partial t} = 0$ and Equation (103) reduces to

$$\frac{dF}{dt} = [F, H]q_{,}p \tag{104}$$

If the variable F under consideration is such that its Poisson bracket with the Hamiltonian of the system vanishes, i.e., $[F, H]q_{,}p = 0$, we get from Equation (104)

$$\frac{dF}{dt} = 0 \tag{105}$$

or
$$F = \text{A constant of motion}$$

We thus find that a dynamical variable of a mechanical system is a constant of motion or an integral of motion, provided that

(*i*) it has no explicit time dependence, and

(*ii*) its Poisson bracket with the Hamiltonian of the system vanishes.

NOTES

6.5.2 Canonical Transformation and Poisson Bracket

Let us consider a mechanical system of *s* degrees of freedom described by generalized coordinates $q_1, ..., q_s$ and generalized momenta $p_1, ..., p_s$. Consider two dynamical variables *f* anf *g* which are functions of the *q*'s and *p*'s. The Poisson bracket of *f* anf *g* is, by definition,

$$[f, g]_{q,p} = \sum_k \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right) \quad (106)$$

Let us consider a canonical transformation of the variables *q*'s and *p*'s, respectively to *Q*'s and *P*'s. In terms of the transformed variables, the Poisson bracket of *f* and *g* is

$$[f, g]_{Q,P} = \sum_k \left(\frac{\partial f}{\partial Q_k} \frac{\partial g}{\partial P_k} - \frac{\partial f}{\partial P_k} \frac{\partial g}{\partial Q_k} \right) \quad (107)$$

The Poisson bracket given by Equation (107) can alternatively be written as

$$[f, g]_{Q,P} = \sum_{k,j} \left[\frac{\partial f}{\partial Q_k} \left(\frac{\partial g}{\partial q_j} \frac{\partial q_j}{\partial P_k} + \frac{\partial g}{\partial p_j} \frac{\partial p_j}{\partial P_k} \right) - \frac{\partial f}{\partial P_k} \left(\frac{\partial g}{\partial q_j} \frac{\partial q_j}{\partial Q_k} + \frac{\partial g}{\partial p_j} \frac{\partial p_j}{\partial Q_k} \right) \right] \quad (108)$$

On rearranging the terms, the above becomes

$$[f, g]_{Q,P} = \sum_k \left\{ \frac{\partial g}{\partial q_j} [f, q_j]_{Q,P} + \frac{\partial g}{\partial p_j} [f, p_j]_{Q,P} \right\} \quad (109)$$

Replacing *f* by *qj* and *g* by *f* we obtain

$$[f, q_j]_{Q,P} = -[q_j, f]_{Q,P} = -\sum_i \left\{ \frac{\partial g_i}{\partial Q_i} \frac{\partial f}{\partial P_k} - \frac{\partial q_i}{\partial P_i} \frac{\partial f}{\partial Q_i} \right\}$$

$$= -\sum_{k,i} \left\{ \frac{\partial q_j}{\partial Q_i} \left(\frac{\partial f}{\partial q_k} \frac{\partial q_k}{\partial P_i} + \frac{\partial f}{\partial p_k} \frac{\partial p_k}{\partial P_i} \right) - \frac{\partial q_j}{\partial P_i} \left(\frac{\partial f}{\partial q_k} \frac{\partial q_k}{\partial Q_i} + \frac{\partial f}{\partial p_k} \frac{\partial p_k}{\partial Q_i} \right) \right\}$$

$$= -\sum_k \left\{ \frac{\partial f}{\partial q_k} \sum_i \left(\frac{\partial q_j}{\partial Q_i} \frac{\partial q_k}{\partial P_i} - \frac{\partial q_j}{\partial P_i} \frac{\partial q_k}{\partial Q_i} \right) + \frac{\partial f}{\partial p_k} \sum_i \left(\frac{\partial q_j}{\partial Q_i} \frac{\partial p_k}{\partial P_i} - \frac{\partial q_j}{\partial P_i} \frac{\partial p_k}{\partial Q_i} \right) \right\}$$

$$= -\sum_k \left\{ \frac{\partial f}{\partial q_k} [q_j, q_k]_{Q,P} + \frac{\partial f}{\partial p_k} [q_j, p_k]_{Q,P} \right\}$$

$$= -\sum_k \frac{\partial f}{\partial p_k} \delta_{jk}$$

NOTES

using the properties $[q_i, q_k] = 0$, $\left[q_j, p_k\right] = 0$ if $j \neq k$, $[q_i, p_k] = 1$ if $j = k$]

or
$$\left[f, q_j\right]_{Q,P} = -\frac{\partial f}{\partial p_j} \tag{110}$$

Similarly, we get

$$\left[f, p_j\right]_{Q,P} = -\frac{\partial f}{\partial q_j} \tag{111}$$

Using Equations (110) and (111) in Equation (108), we get

$$[f, g]_{Q,P} = \sum_j \left(-\frac{\partial g}{\partial q_j}\frac{\partial f}{\partial p_j} + \frac{\partial g}{\partial p_j}\frac{\partial f}{\partial q_j} \right)$$

or
$$[f, g]_{Q,P} = \sum_j \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right) \tag{112}$$

or
$$[f, g]_{Q,P} = [f, g]_{q,p} \tag{113}$$

Thus, the Poisson bracket remains invariant under canonical transformation.

Check Your Progress

12. Define Lagrange brackets.
13. Write the equations of fundamental or basic Poisson brackets.
14. What do you mean by Poisson brackets?

6.6 CALCULUS OF VARIATIONS

For any moving system, the differential of a quantity, say α, usually written as $d\alpha$ refers to an infinitesimal change in the quantity along the path of motion of the system.

Clearly, a differential is determined from the equations of motion that holds for the system. The term **variation** of the quantity, usually written as $\delta\alpha$, on the other hand, corresponds to a transfer of the quantity at a particular time instant when we switch over from one path of motion to another lying close to the former path and is compatible with the constraints imposed on the system. It may be noted that a variation is subject only to the constraints and is otherwise arbitrary.

6.6.1 Calculus of Variations

Let us begin with a simple problem: What is the shortest path between two points in a plane?

The method that will be used to derive it can also be applied to less trivial examples, for instance, to find the shortest path between two points on a curved surface.

Let us take the two points on the plane having coordinates (x_0, y_0) and (x_1, y_1). Any curve that joins them is represented by an equation

$$y = y(x)$$

such that the function $y(x)$ satisfies the boundary conditions

$$y(x_0) = y_0 \quad \text{and} \quad y(x_1) = y_1 \tag{114}$$

Consider two neighboring points on this curve. The distance dl between them is given by

$$dl = \left[dx^2 + dy^2\right]^{\frac{1}{2}} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}} dx$$

or $$dl = \left[1 + y'^2\right]^{\frac{1}{2}} dx$$

where $$y' = \frac{dy}{dx}$$

The total length of the curve joining the two points is

$$l = \int_{x_o}^{x_1} \sqrt{1 + y'^2}\, dx \tag{115}$$

The problem thus reduces to find the function $y(x)$ subject to the boundary conditions given by Equation (114) which will make the above integral the minimum. The problem differs from the more familiar kind of minimum value problem in that what we have to vary in this problem is not just a single variable or a set of variables but a function $y(x)$. However, we can still apply the same criterion: When the integral has a minimum value it must be unchanged to the first order by making a small variation in the function $y(x)$ (we shall not be concerned with the problem of distinguishing maxima from minima. All we shall do is to find the stationary or the extremum values).

More generally, we may be interested in finding the stationary values of an integral of the form

$$l = \int_{x_o}^{x_1} f(y, y')\, dx \tag{116}$$

where $f(y, y')$ is a specified function of y and its first derivative. Let us first solve this general problem.

NOTES

Consider a small variation $\delta y(x)$ in the function $y(x)$ subject to the condition that the values of y at the two end points are unchanged [refer to Figure 6.2].

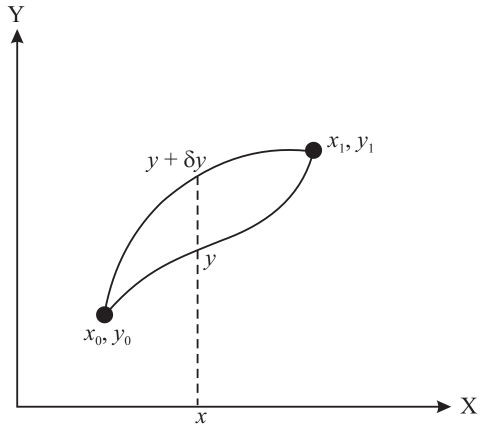

Fig. 6.2 *Variation in Function*

$$\delta y(x_0) = 0; \delta y(x_1) = 0 \tag{117}$$

To the first order, the variation in $f(y, y')$ is

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y'$$

where $$\delta y' = \frac{d(\delta y}{dx}$$

Thus, the variation of the integral I is

$$\delta I = \int_{x_o}^{x_1}\left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\frac{d(\delta y)}{dx}\right)dx$$

The second term may be integrated by parts. The integrated term, namely

$$\left\{\frac{\partial f}{\partial y'}\delta y\right\}_{x_o}^{x_1}$$

vanishes because of the conditions given by Equation (117). Hence, we obtain

$$\delta I = \int_{x_o}^{x_1}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y(x)\,dx \tag{118}$$

Now in order that I be stationary, the variation δI must vanish for an arbitrary small variation $\delta y(x)$ (subject only to the boundary condition given by Equation (117)). This is possible only if the integrand in Equation (118) vanishes identically. Thus, we require

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \tag{119}$$

Equation (119) is known as the Euler-Lagrange equation. It is, in general, a second-order differential equation for the unknown function $y(x)$ whose solution contains two arbitrary constants that may be determined from the boundary conditions.

We can now return to solving the problem we started with—that of finding the curves of shortest length between two fixed points called the **geodesics on a plane**.

Geodesics on a Plane

What is the shortest path between two given points in a plane?

In this case, comparing Equation (115) with Equation (116) we have to choose

$$f = \sqrt{1+y'^2}$$

so that
$$\frac{\partial f}{\partial y} = 0, \qquad \frac{\partial f}{\partial y'} = \frac{1}{2}\left(1+y'^2\right)^{-\frac{1}{2}} 2y' = \frac{y'}{\sqrt{1+y'^2}}$$

Thus, the Euler-Lagrange Equation (119) reads

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

This equation states that the expression inside the parenthesis is a constant and hence y' is a constant. Its solutions are the straight lines

$$y = ax + b \text{ } (a, b \text{ are constants})$$

We have thus proved that the shortest path between two points in a plane is a straight line.

The constants a and b are fixed by the conditions given by Equation (114). So far we have used x as the independent variable, but in the applications to be considered later, we shall be concerned instead with functions of time t. It is easy to generalize the discussion to the case of a function f of s variables $q_1, q_2, \ldots, qs$ and their time derivatives $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s$. In order that the integral

$$I = \int_{t_o}^{t_1} f\left(q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s\right) dt$$

be stationary, it must be unchanged in the first order by a variation in any one of the functions $q_i(t)\left(i = 1, 2, \ldots, s\right)$, subject to the conditions $\delta q_i \; t_o \; = 0 = \delta q_i \; t_1$. Thus, we require Euler-Lagrange equations

$$\frac{\partial f}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) = 0; \quad i = 1, 2, \ldots, s \tag{120}$$

These s number of second-order partial differential equations determine the s functions $qi(t)$ to within $2s$ arbitrary constants of integration.

6.6.2 Variational Principle and Euler-Lagrange Equation

NOTES

The D'Alembert's principle, which is widely used together with Newton's laws of motion for dealing with mechanical systems, is a differential principle. This is because, in using this principle, we need to consider the instantaneous state of a system (defined by positions and velocities in the configuration space of the system) along with some infinitesimal virtual displacements from the instantaneous position.

The variational principle finds immense usefulness in treating mechanical system on the one hand, while on the other hand, it considers the motion of the system as a whole between the given time limits along some small variation in the motion of the system between the same time limits from the actual motion. In this sense, the variational principle is essentially an integral principle. In the following, we will discuss some important aspects of the calculus of variation that happens to be useful for future development of different formulations of mechanics.

Consider a curve given by

$$y = y(x) \tag{121}$$

defined between two points (x_1, y_1) and (x_2, y_2) as shown in Figure 6.3. We may conveniently call the two points as end points.

Let a function $f = f\left(y, \frac{dy}{dx}, x\right) = f\ \ y, y'\ x$ be defined on the above curve.

Our basic problem is to obtain the curve for which the line integral of the function f between the end points (x_1, y_1) and (x_2, y_2) is stationary, i.e.,

$$I = \int_{x_1}^{x_2} f\ \ y, y', x\ \ dx = \text{Extremum (either maximum or minimum)}$$

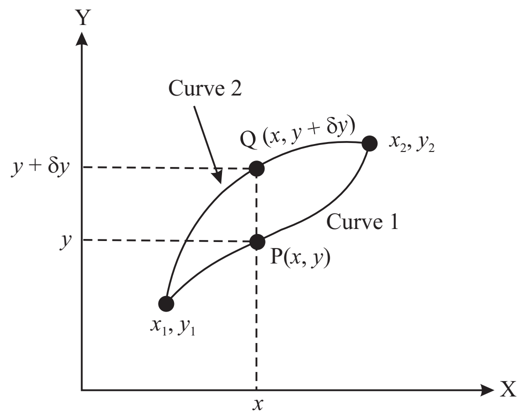

***Fig. 6.3** Neighbouring Curves*

In Figure 6.3, two neighboring curves governed by Equation (121) are shown between the end points $y(x_1) = y_1$ and $y(x_2) = y_2$, the curve-1 corresponding to the stationary value of the integral I. Consider the point $P(x, y)$ on Curve 1 for $x = x$.

NOTES

The point on Curve 2 for the same value of x is the point $Q\ x, y + \delta y$. Here, δy defines the variation in y as we go over from Curve 1 to Curve 2, keeping x the same.

It is useful to associate some parameter, say α, with all the possible curves determined by the constraints between the end points indicated. The α should be such that for some given value, for simplicity, say, for $\alpha = 0$ the Curve 2 coincides with Curve 1.

Corresponding to the extremum value of the integral, y is then a function of both the independent variable x and the parameter α. We may express

$$y(\alpha, x) = y(x) + \alpha\eta(x) \tag{122}$$

where $\eta(x)$ is a function of x which has continuous first derivative and vanishes at the two end points. Clearly, $y(\alpha, x)$ reduces to $y(x)$ at the two end points.

In view of our considerations above, the integral I becomes a function of the parameter α and we get

$$I(\alpha) = \int_{x_1}^{x_2} f\left[y\ \alpha, x\ ,\ y'\ \alpha, x\ ,\ x\right] dx \tag{123}$$

Condition that $I(\alpha)$ has an extremum value is thus

$$\left|\frac{\partial I\ \alpha}{\partial\alpha}\right|_{\alpha=0} = 0 \tag{124}$$

Differentiating Equation (123) with respect to α we obtain

$$\frac{\partial I(\alpha)}{\partial\alpha} = \frac{\partial}{\partial\alpha}\left[\int_{x_1}^{x_2} f\left[y(\alpha, x), y'(\alpha, x), x\right]\right] dx$$

$$= \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial\alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial\alpha}\right] dx \quad \left(\text{we may note that } \frac{\partial x}{\partial\alpha} = 0\right)$$

or

$$\frac{\partial I(\alpha)}{\partial\alpha} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial\alpha} + \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial\alpha\partial x}\right] dx \tag{125}$$

Integrating by parts, we have

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\frac{d}{dx}\left(\frac{\partial y}{\partial x}\right) dx = \left\{\frac{\partial f}{\partial y'}\frac{\partial y}{\partial x}\right\}_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial x} dx \tag{126}$$

Clearly,

$$\left.\frac{\partial y}{\partial\alpha}\right|_{x_1}^{x_2} = \eta(x)\Big|_{x_1}^{x_2} = \eta(x_2) - \eta(x_1) = 0 \tag{127}$$

NOTES

We thus get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \frac{d}{dx}\left(\frac{\partial y}{\partial x}\right) dx = \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha} dx \tag{128}$$

Using Equation (128) in Equation (125) we obtain

$$\frac{\partial I(\alpha)}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha}\right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \eta(x)\, dx \tag{129}$$

We may note that the functions y and y' with respect to which the derivatives of the function f appear on the right hand side of Equation (129) are functions of α. However, for $\alpha = 0$ we get $y(\alpha, x) = y(x)$, $y'(\alpha, x) = y'(x)$ and Equation (129) becomes independent of α. Since $\eta(x)$ is an arbitrary function, for $\left|\frac{\partial I(\alpha)}{\partial \alpha}\right|_{\alpha=0}$ to vanish so that $I(\alpha)$ has an extremum value, we find from Equation (129)

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

or

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y} \tag{130}$$

Equation (130) is the Euler-Lagrange equation as obtained earlier.

The Euler-Lagrange equation can be generalized to the case when

$$f = f\left(y_1, ..., y_s, y'_1, ..., y'_s, x\right) \tag{131}$$

In this case, the equation reads

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'_k}\right) = \frac{\partial f}{\partial y_k}; \qquad k = 1, ..., s \tag{132}$$

The results of the calculus of variation can be expressed in terms of δ-notation as

$$\delta I = \delta \int_{x_1}^{x_2} f\left(y_1, ..., y_s, y'_1, ..., y'_s, x\right) dx = 0 \tag{133}$$

Check Your Progress

15. What do you understand by variation?
16. Define geodesics on a plane.

6.7 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

NOTES

1. In the Lagrangian formulation of mechanics, for describing the motion of a system having s degrees of freedom, the system at any instant of time t is represented by a point in an abstract s-dimensional mathematical space called the configuration space of the system. The point is called the system point at the instant t.
2. In the Hamiltonian formulation, s generalized coordinates and s generalized momenta are independent variables for the system. An abstract s dimensional mathematical space, any point of which gives the s momenta of the system, is called the momentum space of the system.
3. Any point representing the state of motion moves in the momentum space. The curve traced out by the point is called hodograph.
4. As time passes, the point representing the configuration and the state of motion of the system in the phase space traces out a trajectory called the phase trajectory.
5. The function F may have the following forms:

   (i) $F_1(q, Q, t)$, (ii) $F_2(q, P, t)$, (iii) $F_3(p, Q, t)$, (iv) $F_4(p, P, t)$.
6. $K = H + \dfrac{\partial F_1}{\partial t}$
7. $K = H + \dfrac{\partial F_2}{\partial t}$
8. $K = H + \dfrac{\partial F_3}{\partial t}$
9. $K = H + \dfrac{\partial F}{\partial t}$
10. The canonical transformation is said to be an infinitesimal transformation if the new coordinates differ from the old coordinates by infinitesimal amounts and similarly the new momenta differ from the old momenta by infinitesimal amounts.
11. Corresponding to infinitesimal transformations, the transformation equations are of the form

$$\begin{array}{l} Q_k = q_k + \delta q_k \\ P_k = p_k + \delta p_k \end{array} ;\ (k = 1, 2, \ldots\ldots, s)$$

   where δqk and δpk are respectively the infinitesimal changes in the coordinate qk and momentum pk.

12. Consider a mechanical system of s degree of freedom. Let the system be described by generalized coordinates q_1, ….., qs and conjugate momenta p_1, ….., ps.

    Let $f = f(q_1, ..., qs, p_1, ..., ps) = f(q, p)$ and $g = g(q_1, ..., qs, p_1, ...,$ $ps) = g(q, p)$ be two dynamical variables of the system.

    The Lagrange bracket of f and g with respect to the basis (q, p) is written as $\{f, g\}q_, p$ and is defined as

$$\{f, g\}q_, p = \sum_k \left(\frac{\partial q_k}{\partial f} \frac{\partial p_k}{\partial g} - \frac{\partial p_k}{\partial f} \frac{\partial q_k}{\partial g} \right) = 0$$

13. $[qi, qj] = 0 = [pi, pj]$

    $[qi, pj] = \delta ij$

14. The quantity within the parenthesis on the right hand side of the following Equations (87) turns out to be of fundamental importance in the formal development of mechanics and is known as the Poisson bracket (PB) of F and H.

$$\frac{dF}{dt} = \sum_k \frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k} - \sum \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial q_k} + \frac{\partial F}{\partial t}$$

$$\frac{dF}{dt} = \sum_k \left[\frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial q_k} \right] + \frac{\partial F}{\partial t}$$

$$\frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t}$$

15. The term variation of the quantity, usually written as $\delta\alpha$, corresponds to a transfer of the quantity at a particular time instant when we switch over from one path of motion to another lying close to the former path and is compatible with the constraints imposed on the system. It may be noted that a variation is subject only to the constraints and is otherwise arbitrary.
16. The curves of shortest length between two fixed points called the geodesics on a plane.

6.8 SUMMARY

- To describe a function such as the Hamiltonian function $H(q, p, t)$ for the system we need a combination of the configuration space (coordinate space) and the momentum space for the system.
- One important feature of phase trajectory is that no two phase trajectories can intersect with each other.

- As the system develops with time, the point representing the dynamical state called the representative point traces a path or trajectory determined by the Hamilton's canonical equations given by

$$\hat{q}_i = \frac{\partial H}{\partial p_i}, \hat{p}_i = \frac{-\partial H}{\partial p_i} \quad i = 1, \cdots, s$$

- In Lagrangian formulation, there is no restriction to the choice of the generalized coordinates.
- In Hamiltonian formulation, on the other hand, the system is described by not only the s independent generalized coordinates but the description includes s independent generalized momenta $p_1, \ldots.., p_s$ defined through the equations $p_k = \frac{\partial L}{\partial \dot{q}_k}, \quad k = 1, \ldots.., s$.
- In point transformation, the new Lagrangian can be obtained from the old by direct substitution of the transformation relations.
- Canonical transformations are, however, problem independent.

6.9 KEY WORDS

- **Configuration space:** The vector space explained by generalized coordinates is known as the configuration space of the physical system.
- **Hodograph:** The curve traced out by any point in the momentum space representing the state of motion is called hodograph.
- **Canonical transformation:** It is a change of canonical coordinates that protects the form of Hamilton's equations.
- **Generating function:** It is a process of encoding an infinite sequence of numbers by considering them as the coefficients of a power series. This formal power series is known as the generating function.
- **Infinitesimal transformation:** The canonical transformation is said to be an infinitesimal transformation if the new coordinates differ from the old coordinates by infinitesimal amounts and similarly the new momenta differ from the momenta by infinitesimal amounts.
- **Differential:** For any moving system, the differential of a quantity refers to an infinitesimal change in the quantity along the path of motion of the system.
- **Variation:** The term variation of the quantity corresponds to a transfer of the quantity at a particular time instant when we switch over from one path of motion to another lying close to the former path and is compatible with the constraints imposed on the system.
- **Geodesics on a plane:** The curves of shortest length between two fixed points called the Geodesics on a plane.

6.10 SELF ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short-Answer Questions

1. Write a short note on point transformations.
2. Derive generating function of the form F_1 (q, Q, t).
3. Derive generating function of the form F_2 (q, p, t).
4. Derive generating function of the form F_3 (p, Q, t).
5. Derive generating function of the form F_4 (p, p, t).
6. Analyse different conditions for canonical transformations.
7. Write properties of Lagrange brackets.
8. Derive the transformation relations between the old sets of coordinates and momenta to new sets of coordinates and momenta.
9. Describe integrals of motion briefly.
10. Give a brief account of canonical transformation and Poisson brackets.
11. Interpret geodesics on a plane briefly.

Long-Answer Questions

1. Discuss Liouville's theorem.
2. Describe transformation relations for different forms of the generating function.
3. Explain calculus of variations with suitable diagrams.
4. Describe Variational principle deriving Euler-Lagrange equation.

6.11 FURTHER READINGS

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NOTES

BLOCK - III
KINEMATICS OF RIGID BODY MOTION AND SPECIAL THEORY OF REATIVITY

UNIT 7 MOMENT OF INERTIA

Structure

7.0 INTRODUCTION

In physics, moment of inertia is defined as quantitative measure of the rotational inertia of a body. The moment of inertia, also known as the angular mass or rotational inertia, of a rigid body is considered as a tensor that determines the torque required for a desired angular acceleration about a rotational axis. It depends on the body's mass distribution and the axis selected. Principally, the term moment of inertia is specifically given to rotational inertia, the rotational analog of mass for linear motion, hence the moment of inertia must be specified with respect to a selected axis of rotation. Fundamentally, the moment of inertia is a physical quantity which describes how easily a body can be rotated about a given axis. Inertia is the property of matter which resists change in its state of motion. Consequently, inertia is considered as a measure of the force that keeps a stationary object stationary or a moving object moving at its current speed. The larger the inertia, the greater the force that is required to bring some change in its velocity in a given amount of time.

In this unit, you will learn about moment of inertia with reference to moments and products of inertia, moment of inertia of a body about any line through the origin of coordinate frame, the momental ellipsoid and rotation of coordinate axes.

7.1 OBJECTIVES

NOTES

After going through this unit you will be able to:

- Understand the concept of moment of inertia
- Define what products of inertia is
- Explain moment of inertia of a body about any line
- Analyse the moment of inertia through the origin of coordinate frame
- Describe the momental ellipsoid
- Understand the rotation of coordinate axes

7.2 MOMENTS AND PRODUCTS OF INERTIA

Let us consider an element of area dA within an area A in the x-y plane as shown in Figure 7.1 whose centroid is (x, y). Then the term $\int y^2 dA$ is known as moment of inertia of the area about x-axis and is denoted by I_{xx}.

So $$I_{xx} = \int y^2 d\,A$$

Similarly, $$I_{yy} = \int x^2 d\,A\,.$$

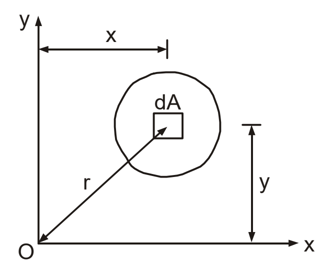

Fig. 7.1 *Inertia*

Polar Moment of Inertia

Moment of inertia about an axis perpendicular to the plane of an area is known as polar moment of inertia. It may be denoted as I_{zz}. Thus, the moment of inertia about an axis perpendicular to the plane area at O in Figure 7.1 is called polar moment of inertia at point O and can be written as

$$I_{zz} = \int r^2 dA$$

Radius of Gyration

Radius of gyration is a mathematical term and is defined by the relation

$$I_{xx} = Ak_x^{\,2} = \int y^2 dA \quad \text{or} \quad k_x = \sqrt{\frac{I_{xx}}{A}}$$

Similarly, $k_y = \sqrt{\frac{I_{yy}}{A}}$ and $k_z = \sqrt{\frac{I_{zz}}{A}}$

where k is known as radius of gyration; I = moment of inertia and A is the cross-sectional area.

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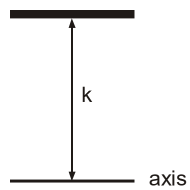

Fig. 7.2 *Radius of Gyration*

From the above relation a geometrical meaning can be assigned to the term radius of gyration. In case of moment of inertia the area is squeezed and kept as a strip of negligible width at a distance k such that there is no change in the moment of inertia. It can be represented as shown in Figure 7.2

7.2.1 Theorems of Moment of Inertia

There are two theorems of moment of inertia:

1. Perpendicular axis theorem.
2. Parallel axis theorem.

1. Perpendicular Axis Theorem: The moment of inertia of an area about an axis perpendicular to its plane (polar moment of inertia) at any point is equal to the sum of moments of inertia about any two mutually perpendicular axes through the same point and lying in the plane of the area.

Proof: Referring to Figure 7.3. Let *z-z* be the axis perpendicular to the plane of the area. Then we have to prove

$$I_{zz} = I_{xx} + I_{yy}.$$

Let us consider an elemental area dA at a distance r from O. Let the co-ordinates of dA be x and y. Then from the definition,

$$I_{zz} = \int r^2 dA = \int (x^2 + y^2)\, dA = \int x^2\, dA + \int y^2\, dA = I_{xx} + I_{yy}.$$

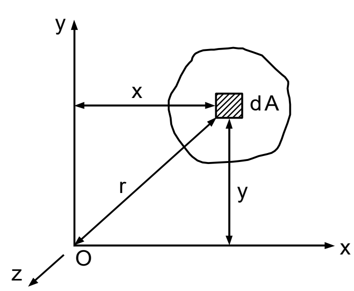

Fig. 7.3 *Perpendicular Axis Theorem*

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2. **Parallel Axis Theorem:** Moment of inertia about any axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and square of the distance between the two parallel axes.

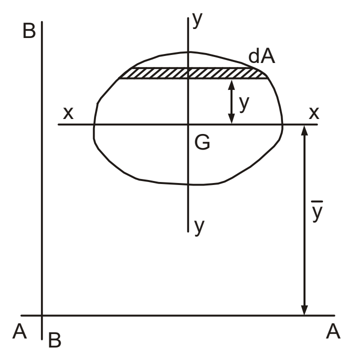

Fig. 7.4 *Parallel Axis Theorem*

Proof: Let us consider an elemental strip of area *dA* at a distance *y* from *x*-axis and parallel to it. So, its distance from the axis *AA* will be $(\bar{y} + y)$, according to the Figure 7.4.

Moment of inertia of the elemental component about axis *AA* will be

$$= dA\,(y + \bar{y})^2$$

So, the moment of inertia of the entire lamina about *AA* axis

$$I_{AA} = \int (y + \bar{y})^2\, dA$$

$$= \int y^2 dA + \int 2y\,\bar{y}\; dA + \int \bar{y}^2 dA$$

$$= \int y^2 dA + 2\bar{y} \int y\; dA + \bar{y}^2 \int dA$$

Now, $\int y^2 dA$ = Moment of inertia of the area about x–x = I_{xx}.

$\int y\; dA = A.\int \frac{y\; dA}{A}$ = the distance of centroid from the reference axis xx. But xx is passing through the centroid itself. So $\int \frac{y\; dA}{A} = 0$ and hence $2\,\bar{y} \int y\; dA = 0$.

Again $\bar{y}^2 \int dA \quad = A\,\bar{y}^2$

$$\therefore \quad I_{AA} = I_{xx} + A\,\bar{y}^2.$$

Note: The above equation cannot be applied to any two parallel axes. One of the axes must be centroidal axis.

7.2.2 Mass Moment of Inertia

Mass moment of inertia of a body about an axis is defined as the sum total of product of its elemental masses and square of their distance from the axis. Thus, the mass moment of the body shown in Figure 7.5 about axis *AB* is given by

$I_{AB} = \int r^2 dm$ where r is the distance of element of mass dm from AB.

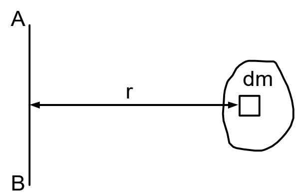

Fig. 7.5 *Mass Moment of Inertia*

NOTES

Note: Mass moment of inertia is similar to the area moment of inertia. In this case instead of area, mass of the body is to be considered.

Moments of Inertia, Products of Inertia and Inertia Tensor of a Rigid Body

As per the standard equation,

$$\vec{J} = \sum_{i=1}^{N} m_i \left[r_i^2 \vec{\Omega} - \left(\vec{\Omega} \cdot \vec{r_i} \right) \vec{r_i} \right]$$

we have

$$\frac{1}{2}\vec{\Omega}\vec{J} = \frac{1}{2}\vec{\Omega} \cdot \sum_{i=1}^{N} m_i \left[r_i^2 \vec{\Omega} - \left(\vec{\Omega} \cdot \vec{r_i} \right) \vec{r_i} \right]$$

$$= \frac{1}{2}\sum m_i r_i^2 \Omega^2 - \frac{1}{2}\sum m_i \left(\vec{\Omega} \cdot \vec{r_i} \right)^2 \quad (1)$$

In view of the above result, we may express the rotational kinetic energy as given by Equation

$$T_{rot} = \sum \frac{1}{2} m_i \left[\Omega^2 r_i^2 - \left(\vec{\Omega} \cdot \vec{r_1} \right)^2 \right]$$

as

$$T_{\text{rot}} = \frac{1}{2}\vec{\Omega} \cdot \vec{J} \quad (2)$$

If Ω_x. Ω_y, Ω_z be, respectively, the Cartesian components of $\vec{\Omega}$ along the X, Y and Z axes and if x_i, y_i, and z_i be the components of $\vec{r_1}$ along these axes, respectively, we may write the rotational kinetic energy given by Equation

$$T_{rot} = \sum \frac{1}{2} m_i \left[\Omega^2 r_i^2 - \left(\vec{\Omega} \cdot \vec{r_1} \right)^2 \right]$$

NOTES

as

$$T_{\text{rot}} = \frac{1}{2}\sum m_i\left(\Omega_x^2+\Omega_y^2+\Omega_z^2\right)\left(x_i^2+y_i^2+z_i^2\right) - \frac{1}{2}\sum m_i\left(\Omega_x x_i+\Omega_y y_i+\Omega_z z_i\right)^2$$

$$= \frac{1}{2}\sum m_i\left[\Omega_x^2 x_i^2+\Omega_x^2 y_i^2+\Omega_x^2 z_i^2+\Omega_y^2 x_i^2+\Omega_y^2 y_i^2+\Omega_y^2 z_i^2+\Omega_z^2 x_i^2+\Omega_z^2 y_i^2+\Omega_z^2 z_i^2\right]$$

$$-\frac{1}{2}\sum m_i\left[\Omega_x^2 x_i^2+\Omega_y^2 y_i^2+\Omega_z^2 z_i^2+2\Omega_x\Omega_y x_i y_i+2\Omega_y\Omega_z y_i z_i+2\Omega_z\Omega_x z_i x_i\right]$$

Rearranging the terms and simplifying, we may write the above as

$$T_{\text{rot}} = \frac{1}{2}\sum m_i\left[\Omega_x^2\left(y_i^2+z_i^2\right)+\Omega_y^2\left(z_i^2+x_i^2\right)+\Omega_z^2\left(x_i^2+y_i^2\right)\right]$$

$$-\frac{1}{2}\sum m_i\left[\Omega_x\Omega_y x_i y_i+\Omega_y\Omega_x y_i x_i+\Omega_y\Omega_z y_i z_i+\Omega_z\Omega_y z_i y_i+\Omega_z\Omega_x z_i x_i+\Omega_x\Omega_z x_i z_i\right] \quad (3)$$

Defining

$$I_{xx} = \sum m_i\left(y_i^2+z_i^2\right)=\sum m_i\left(r_i^2-x_i^2\right)$$

$$I_{yy} = \sum m_i\left(x_i^2+z_i^2\right)=\sum m_i\left(r_i^2-y_i^2\right) \quad (4)$$

$$I_{zz} = \sum m_i\left(x_i^2+y_i^2\right)=\sum m_i\left(r_i^2-z_i^2\right)$$

and

$$I_{\text{xy}} = -\sum m_i\,.\,x_i\,.\,y_i$$

$$I_{\text{yz}} = -\sum m_i\,.\,y_i\,.\,z_i \quad (5)$$

$$I_{\text{zx}} = -\sum m_i\,.\,z_i\,.\,x_i$$

we may rewrite the rotational kinetic energy of the rigid body given by Equation (3) as

$$T_{\text{rot}} = \frac{1}{2}\left[\Omega_x^2 I_{xx}+\Omega_y^2 I_{yy}+\Omega_z^2 I_{zz}\right]+\Omega_x\Omega_y I_{xy}+\Omega_y\Omega_z I_{yz}+\Omega_z\Omega_x I_{zx} \quad (6)$$

All the quantities defined by Equation (4) and (5) have the dimensions of moment of inertia.

$I_{\text{xx}}, I_{\text{yy}}, I_{\text{zz}}$ given by Equation (4) are called the moments of inertia.

$I_{\text{xy}}, I_{\text{yz}}, I_{\text{zx}}$ given by Equation (5) are called the products of inertia.

Now, we may write Equation (6) in a shorter form as

$$T_{\text{rot}} = \frac{1}{2}\sum_{j,k} I_{jk}\Omega_j\Omega_k \quad (7)$$

where $\qquad j,\ k = x, y, z \qquad (8)$

I_{jk} as introduced above is called **inertia tensor of the rigid body**.

From the definitions of moments and products of inertia given by Equations (4) and (5), we find that

$$I_{jk} = I_{kj} \tag{9}$$

NOTES

Clearly, the **inertia tensor of the rigid body is a symmetric tensor**.

If we denote x, y, z by x_1, x_2, x_3, respectively, then, in general, any element of the inertia tensor is given by

$$I_{jk} = I_{kj} = \sum_{i=1}^{N} m_i \left[r_i^2 \delta_{jk} - x_{ij} x_{ik} \right];\ j,\ k = 1,\ 2,\ 3. \tag{10}$$

In case the rigid body is a continuous mass distribution, the summation signs in Equation (4) and (5) should be replaced by integration, so as to get

$$I_{xx} = \int \left(r^2 - x^2\right) \rho(r)\, dV = \int \rho(r) \left(y^2 + z^2\right) dV$$

$$I_{yy} = \int \left(r^2 - y^2\right) \rho(r)\, dV = \int \rho(r) \left(x^2 + z^2\right) dV \tag{11}$$

$$I_{zz} = \int \left(r^2 - z^2\right) \rho(r)\, dV = \int \rho(r) \left(x^2 + y^2\right) dV$$

$$I_{xy} = -\int \rho(r)\, xy dV$$

and

$$I_{yz} = -\int \rho(r)\, yz dV \tag{12}$$

$$I_{zx} = -\int \rho(r)\, zx dV$$

In general, we obtain

$$I_{jk} = \int \rho(r) \left(r^2 \delta_{jk} - x_j x_k\right) dV \tag{13}$$

In terms of the inertia tensor and angular velocity components, the total kinetic energy of the rigid body can be expressed as

$$T = \frac{1}{2} M V^2 + \frac{1}{2} \sum_{j,k} I_{jk} \Omega_j \Omega_k \tag{14}$$

We may express the angular momentum components in terms of moments of inertia and products of inertia. We get from Equation

$$\vec{J} = \sum_{i=1}^{N} m_i \left[r_i^2 \vec{\Omega} - \left(\vec{\Omega} \cdot \vec{r_i} \right) \vec{r_i} \right]$$

the x-component of $\vec{J}$ as

$$J_x = \sum_{i=1}^{N} m_i \left[r_1^2 \Omega_x - \left(\Omega_x x_i + \Omega_y y_i + \Omega_z z_i \right) x_i \right]$$

$$= \Omega_x \sum m_i \left(r_i^2 - x_1^2 \right) - \Omega_y \sum m_i y_i x_i - \Omega_z \sum m_i z_i x_i$$

Using Equation (4) and (5) in the above, we get

$$J_x = I_{xx}\Omega_x + I_{xy}\Omega_y + I_{xz}\Omega_z \qquad (15)$$

Similarly, we obtain

NOTES

$$J_y = I_{yx}\Omega_x + I_{yy}\Omega_y + I_{yz}\Omega_z \qquad (16)$$

$$J_z = I_{zx}\Omega_x + I_{zy}\Omega_y + I_{zz}\Omega_z \qquad (17)$$

In general, we may write for any component of the angular momentum

$$J_j = \sum_k I_{jk}\Omega_k \qquad (18)$$

where $$j,\ k = x, y, z \qquad (19)$$

Let U be the potential energy of the rigid body. It is, in general, a function of six generalized coordinates which define the configuration of the body in space.

The Lagrangian function of the rigid body is then

$$L = T - U$$

$$L = \frac{1}{2}MV^2 + \frac{1}{2}\sum_{j,k} I_{jk}\Omega_j\Omega_k - U \qquad (20)$$

Example 7.1 Determine the moment of inertia of a rectangle of sides a and b about the centroidal axes and also the axis AB as shown in Figure 7.6.

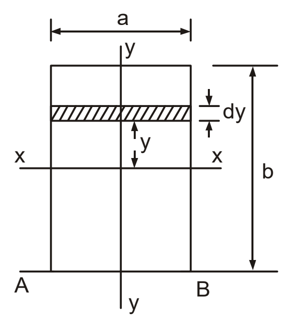

Fig. 7.6

Solution: Let us consider an elemental strip of width dy at a distance y from the centroidal axis xx. So the moment of inertia of the area about the centroidal axis xx will be

$$I_{xx} = \int_{-b/2}^{b/2} y^2 dA = \int_{-b/2}^{b/2} y^2 \times a\,dy \qquad [\because\ dA = a\,dy]$$

$$= a\left[\frac{y^3}{3}\right]_{-b/2}^{b/2} = a\left[\frac{b^3}{24} + \frac{b^3}{24}\right] = \frac{ab^3}{12}$$

Similarly, $$I_{yy} = \int_{-a/2}^{a/2} x^2 \times b\,dx = b\left[\frac{x^3}{3}\right]_{a/2}^{a/2} = \frac{ba^3}{12}.$$

$$\therefore \quad I_{AB} = I_{xx} + A\bar{y}^2 = \frac{ab^3}{12} + ab \times \left(\frac{b}{2}\right)^2$$

$$= \frac{ab^3}{12} + \frac{ab^3}{4} = \frac{ab^3 + 3ab^3}{12} = \frac{ab^3}{3}.$$

NOTES

Note: The moment of inertia of a hollow rectangular area as shown in Figure 7.7 will be

$$I_{xx} = \frac{mn^3}{12} - \frac{ab^3}{12} \text{ and } I_{yy} = \frac{nm^3}{12} - \frac{ba^3}{12}.$$

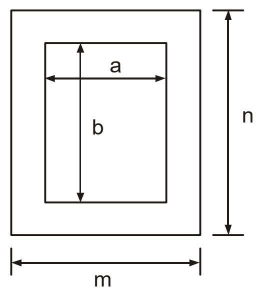

Fig. 7.7

Example 7.2 Determine the moment of inertia of a triangle of base b and height h about the axes OX and OY and also about the centroidal axes, as shown in Figure 7.8.

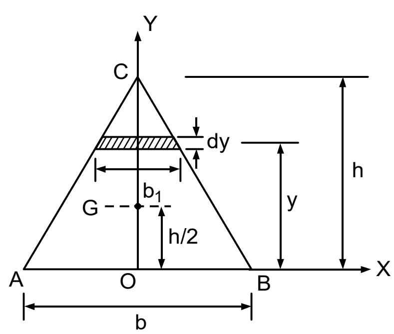

Fig. 7.8

Solution: Let us consider an elemental strip at a distance y from the base AB. Let dy be the thickness of the strip and dA its area. Width of this strip is given by

$$b_1 = \frac{h-y}{h} \times b.$$

Moment of inertia of this strip about AB (OX)

$$= y^2\, dA = y^2\, b_1\, dy = y^2 \frac{(h-y)}{h} \times b \times dy$$

NOTES

$\therefore$ Moment of inertia of the triangle about AB (OX)

$$I_{OX} = \int_0^h y^2 \frac{(h-y)}{h} \times b \times dy = \int_0^h \left(y^2 - \frac{y^3}{h} \right) b\,dy$$

$$= b\left[\frac{y^3}{3} - \frac{y^4}{4h} \right]_0^h = b\left[\frac{h^3}{3} - \frac{h^4}{4h} \right]$$

$$I_{OX} = \frac{bh^3}{12}.$$

Similarly, the moment of inertia of the ΔOBC about OY

$$= \frac{1}{12} h \left(\frac{b}{2} \right)^3 = \frac{1}{96} hb^3$$

Hence, M.O.I. of the ΔABC about

$$\therefore \quad I_{OY} = 2 \times \frac{1}{96} hb^3 = \frac{1}{48} hb^3.$$

It is noted that the triangle ABC is symmetrical about OY.

Moment of Inertia about the Centroidal Axis GG: From the parallel axis theorem

$$I_{AB} = I_{GG} + A\bar{y}^2$$

Now, $\bar{y}$ the distance between the non-centroidal axis AB and centroidal axis GG is equal to $h/3$

or $$\frac{bh^3}{12} = I_{GG} + \frac{1}{2} bh \left(\frac{h}{3} \right)^2 = I_{GG} + \frac{bh^3}{18}$$

$$\therefore \quad I_{GG} = \frac{bh^3}{12} - \frac{bh^3}{18} = \frac{bh^3}{36}.$$

Example 7.3 Determine the moment of inertia of a circle of diameter d about its diametral axis.

Solution: Let us consider an element of sides $r\,d\theta$ and dr as shown in the figure. Moment of inertia of the element about the diametral axis x-x

$$= y^2\,dA$$
$$= (r \sin \theta)^2\, r\,d\theta\,dr$$
$$= r^3 \sin^2 \theta\, d\theta\, dr.$$

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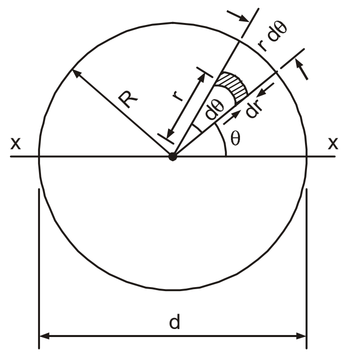

Fig. 7.9

∴ Moment of inertia of the circle about x-x axis given by

$$I_{xx} = \int_0^R \int_0^{2\pi} r^3 \sin^2 \theta \, d\theta \, dr$$

$$= \int_0^R \int_0^{2\pi} r^3 \frac{(1-\cos 2\theta)}{2} d\theta \, dr$$

$$= \int_0^R \frac{r^3}{2} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi} dr$$

$$= \left[\frac{r^4}{8} \right]_0^R [2\pi - 0 + 0 - 0] = \frac{2\pi R^4}{8}$$

$$\therefore \quad I_{xx} = \frac{\pi R^4}{4} = 0.78\,5R^4$$

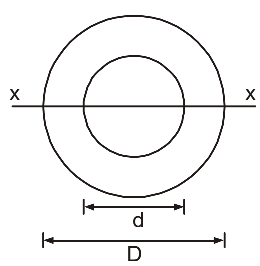

Fig. 7.10

If d is the diameter of the circle, then $R = d/2$

$$\therefore \quad I_{xx} = \frac{\pi}{4}\left(\frac{d}{2}\right)^4$$

$$= \frac{\pi d^4}{64} = 0.049\, d^4.$$

Note: Refering to the Figure 7.10. Moment of inertia of the hollow circle about xx

$$I_{xx} = \frac{\pi D^4}{64} - \frac{\pi d^4}{64} = \frac{\pi}{64}(D^4 - d^4).$$

Example 7.5 Determine the moment of inertia of a quarter of a circle about the base. Also determine the moment of inertia the centroidal x-x axis.

Solution: Let us consider an element of sides r and $r\,d\theta$ as shown in Figure 7.11.

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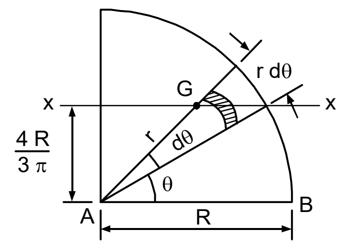

Fig. 7.11

Moment of the element about AB

$$= (r \sin\theta)^2 \,.\, r\,d\theta\,dr$$
$$= r^3 \sin^2\theta\,d\theta\,dr$$
$$= \frac{r^3}{2}(1-\cos 2\theta)\,d\theta\,dr$$

$\therefore$ Moment of inertia of the area about AB

$$I_{AB} = \int_0^R\int_0^{\pi/2} \frac{r^3}{2}(1-\cos 2\theta)\,d\theta\,dr$$
$$= \int_0^R \frac{r^3}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} dr = \left[\frac{r^4}{8}\right]_0^R \left[\frac{\pi}{2} - 0\right]$$
$$= \frac{\pi R^4}{16}.$$

Now for the moment of inertia of the area about centroidal x-x axis

We know $I_{AB} = I_{xx} + A\bar{y}^2$

or $$\frac{\pi R^4}{16} = I_{xx} + \frac{\pi R^2}{4} \times \left(\frac{4R}{3\pi}\right)^2$$

or $$I_{xx} = \frac{\pi R^4}{16} - \frac{4R^4}{9\pi} = 0.055\,R^4.$$

Example 7.6 Determine the moment of inertia with respect to x-x axis for the area enclosed by the ellipse whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

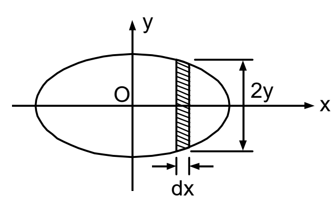

Fig. 7.12

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Solution: Let us consider an element of area $dA = 2y \cdot dx$

Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

or $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$

or $y = b\sqrt{1 - \frac{x^2}{a^2}}$.

$\therefore\ dA = 2b\sqrt{1 - \frac{x^2}{a^2}}\, dx = \frac{2b}{a}\sqrt{a^2 - x^2}\, dx$

$$\therefore\ I_{xx} = \int_{-a}^{a} y^2 dA = \int_{-a}^{a} \frac{b^2}{a^2}(a^2 - x^2).\frac{2b}{a}.\sqrt{a^2 - x^2}\, dx = \frac{2b^3}{a^3}\int_{-a}^{a}(a^2 - x^2)^{3/2} dx$$

Let $x = a \sin\theta$ or, $dx = a\cos\theta\, d\theta$

at $x = a;\ \theta = \frac{\pi}{2}$

at $x = -a;\ \theta = -\frac{\pi}{2}$

$$\therefore\quad I_{xx} = \frac{2b^3}{a^3}\int_{-\pi/2}^{\pi/2} a^3\cos^3\theta . a\cos\theta\, d\theta$$

$$= 2ab^3\int_{-\pi/2}^{\pi/2}\cos^4\theta\, d\theta = 2ab^3 \times \frac{\pi}{8} = \frac{8ab^3}{4}.$$

Check Your Progress

1. What do you understand by polar moment of inertia?
2. Define radius of gyration.
3. Give definition of the mass moment of inertia.

7.3 MOMENT OF INERTIA OF A BODY ABOUT ANY LINE THROUGH THE ORIGIN OF COORDINATE FRAME

The moment of inertia, also termed as the angular mass or rotational inertia, of a rigid body is a tensor that determines the torque essential for a desired angular acceleration about a rotational axis. It depends on the body's mass distribution and the axis selected. Its simplest definition is the second moment of mass with respect to distance from an axis. The bodies which are constrained to rotate in a plane, only their moment of inertia about an axis perpendicular to the plane is

NOTES

considered. Basically, the mechanics involves transforming observational quantities from one coordinate system to another and will use a special subset called linear transformations for defining the coordinate transformations. Such coordinate transformations relate the coordinates in one frame to those in a second frame by means of a system of linear algebraic equations. Thus, if a vector $\vec{X}$ in one coordinate system has components X_j in a primed-coordinate system a vector $\vec{X}'$ to the same point will have components X_j given by,

$$X_i' = \sum_j A_{ij} X_j + B_i$$

In vector notation we could write this as,

$$\vec{X}' = A\vec{X} + \vec{B}$$

This defines the general class of linear transformation where **A** is some matrix and $\vec{B}$ is a vector. This general linear form may be divided into two constituents, the matrix **A** and the vector $\vec{B}$. It is clear that the vector $\vec{B}$ may be interpreted as a shift in the origin of the coordinate system, while the elements $\mathbf{A}_{ij}$ are the cosines of the angles between the axes X_i and X_j and are called the directions cosines (Refer Figure). Certainly, the vector is a vector $\vec{B}$ from the origin of the un-primed coordinate frame to the origin of the primed coordinate frame. Consider two points that are fixed in space and a vector connecting them, the length and orientation of that vector will be independent of the origin of the coordinate frame in which the measurements are made. Transformations that scaled each coordinate by a constant amount, while linear, would change the length of the vector as measured in the two coordinate systems. Since we are considering the coordinate system to describe the vector, hence its length must be independent of the coordinate system. Thus we will analyse the linear transformations to those that transform orthogonal coordinate systems while preserving the length of the vector.

Thus the matrix A must satisfy the condition,

$$\vec{X}' \bullet \vec{X}' = \left(A\vec{X}\right) \bullet \left(A\vec{X}\right) = \vec{X} \bullet \vec{X}$$

Which in component form becomes,

$$\sum_i \left(\sum_j A_{ij} X_j \right) \left(\sum_k A_{ik} X_k \right) = \sum_j \sum_k \left(\sum_i A_{ij} A_{ik} \right) X_j X_k = \sum_i X_i^2$$

This is true for all vectors in the coordinate system so that,

$$\sum_i A_{ij} A_{ik} = \delta_{jk} = \sum_i A_{ji}^{-1} A_{ik}$$

The Kronecker δ_{ij} deltais the unit matrix and any element of a group that multiplies another and produces that group's unit element is defined as the inverse of that element. Therefore,

$$A_{ji} = \left[A_{ij}\right]^{-1}$$

NOTES

Interchanging the elements of a matrix produces a new matrix which is termed as the transpose of the matrix. Thus orthogonal transformations that preserve the length of vectors have inverses that are simply the transpose of the original matrix so that,

$$A^{-1} = A^{T}$$

Figure 7.13 shows two coordinate frames related by the transformation angles φ_{ij}. Four coordinates are necessary if the frames are not orthogonal.

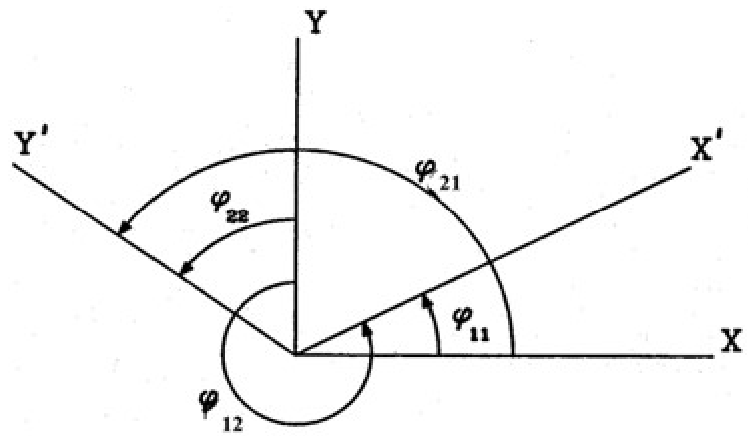

Fig. 7.13 *Coordinate Frames*

7.4 THE MOMENTAL ELLIPSOID

In order to discuss the dynamics of rigid body, equations may be written,

$$\sigma = J\omega, \quad T = \frac{1}{2}(\omega, J\omega) \tag{21}$$

A much more satisfactory definition on an inertial system is provided by the general theory of relativity, in which it is a system in the neighborhood of which the gravitational field vanishes, whether this field is produced by matter or by acceleration or rotation of the axes relative to the coordinate system mentioned above.

with

$$J = \begin{pmatrix} A & -H & -G \\ -H & B & -F \\ -G & -F & C \end{pmatrix} \tag{22}$$

NOTES

As per the transformaion law for the tensor J then becomes

$$\bar{J} = SJS^T \tag{23}$$

The components of J in the space coordinate system are not constant, because in this system the values of the x's change with time. For this reason it is convenient to work in the body coordinate system when discussing the motion of rigid rotators.

The inertia tensor $\bar{J}$ is symmetric and therefore its matrix has three real eigenvalues, and the corresponding eigenvectors are mutually orthogonal. If these three vectors are introduced as the coordinate axes of the body coordinate system, then $\bar{J}$ is diagonal when referred to these axes, i.e., the products of inertia vanish in this coordinate system. The diagonal elements of J are positive for any coordinate system, so that the eigen-values A, B, C of J are positive.

The directions of the eigenvectors of J are called the *principal axes* of the rotator relative to the fixed point. In this system, the kinetic energy can be written:

$$T = \tfrac{1}{2}(A\bar{\omega}_1^2 + B\bar{\omega}_2^2 + C\bar{\omega}_2^2) \tag{24}$$

and the angular momentum has components $(A\bar{\omega}_1, B\bar{\omega}_2, C\bar{\omega}_2)$. If the angular velocity happens to be along one of the principal axes, the angular velocity and angular momentum are therefore parallel to each other.

Associated with a symmetric matrix $\bar{J}$ there is a quadratic form, and a quadric surface defined by

$$(x, Jx) = \bar{J}_{ij}\bar{x}_i\bar{x}_j = 1 \tag{25}$$

which, if J is the inertia matrix, is necessarily an ellipsoid. This surface is called the *momental ellipsoid* because the moment of inertia of the rotator about an axis through the fixed point is easily expressed in terms of this ellipsoid. The moment of inertia I about an axis is defined by the equation.

$$T = \tfrac{1}{2}I\omega^2 \qquad (\omega \text{ along the axis}) \tag{26}$$

If s_p denotes the distance of the pth particle of the rotator from the axis, then

$$I = \sum_p m_p s_p^2 \tag{27}$$

Now let x be parallel to ω. Then, from Equations (25) and (21),

$$\omega = (2T)^{1/2}X$$

Insertion of this in Equation (26) then yields

$$T = \tfrac{1}{2}I(2Tx^2), \qquad I = \frac{1}{x^2}$$

Thus the moment of inertia about an axis is the square of the reciprocal of the distance from the origin to the momental ellipsoid in the direction of the axis.

7.5 ROTATION OF COORDINATE AXES

The defining equations for the moments and products of inertia, do not require that the origin of the Cartesian coordinate system be taken at the mass center. Next, one can calculate the moments and products of inertia for a given body with respect to a set of parallel axes that do not pass through the mass center. Consider the body shown in Figure 7.14. The mass center is located at the origin $O' \equiv C$ of the primed system $x'\, y'\, z'$. The coordinate of O' with respect to the unprimed system xyz is (xc, yc, zc). An infinitesimal volume element dV is located at (x, y, z) in the unprimed system and at (x', y', z') in the primed system. These coordinates are related by the equations,

$$x = x' + x_c, \quad y = y' + y_c, \quad z = z' + z_c. \tag{28}$$

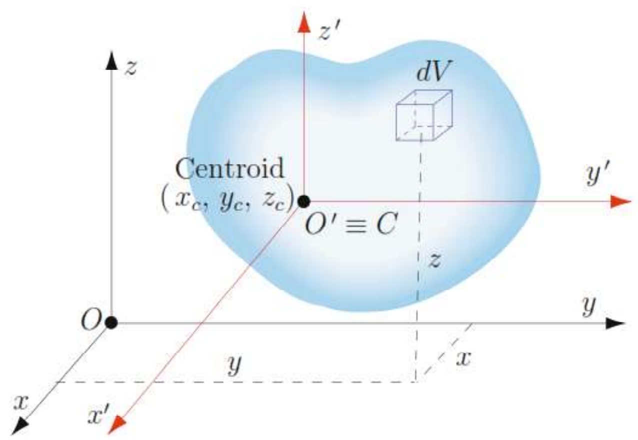

Fig. 7.14 *Rigid Body and Centroidal Axes $x'y'z'$: $x = x' + x_c$, $y = y' + y_c$, $z = z' + z_c$*

The moment of inertia about the x-axis can be written in terms of primed coordinates

$$\begin{aligned} I_{xx} &= \int_V \rho \left[(y' + y_c)^2 + (z' + z_c)^2 \right] \mathrm{d}V \\ &= I_{Cx'x'} + 2y_c \int_V \rho y' \mathrm{d}V + 2z_c \int_V \rho z' \mathrm{d}V + m\left(y_c^2 + z_c^2\right), \end{aligned} \tag{29}$$

where m is the total mass of the rigid body, and the origin of the primed coordinate system was chosen at the mass center. One can write

$$\int_V \rho x' \mathrm{d}V = \int_V \rho y' \mathrm{d}V = \int_V \rho z' \mathrm{d}V = 0, \tag{30}$$

NOTES

and therefore, the two integrals on the right-hand side of (29) are zero. In a similar way, one can obtain I_{yy} and I. The results are summarized as follows:

$$\begin{aligned} I_{xx} &= I_{Cx'x'} + m\left(y_c^2 + z_c^2\right), \\ I_{yy} &= I_{Cy'y'} + m\left(x_c^2 + z_c^2\right), \\ I_{zz} &= I_{Cz'z'} + m\left(x_c^2 + y_c^2\right). \end{aligned} \qquad (31)$$

or, in general,

$$I_{kk} = I_{Ck'k'} + md^2, \qquad (32)$$

where d is the distance between a given unprimed axis and a parallel primed axis passing through the mass center C. Equation (32) represents the *parallel – axes*

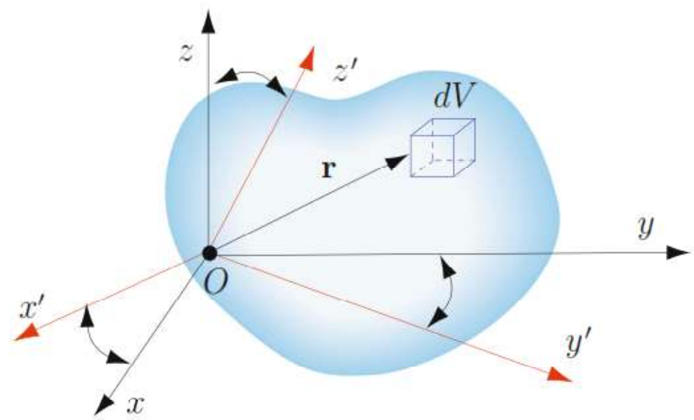

Fig. 7.15 *Rotation of Coordinate Axes*

Theorem. The products of inertia are obtained in a similar manner,

$$I_{xy} = \int_V \rho\left(x' + x_c\right)\left(y' + y_c\right) \mathrm{d}V$$

$$= I_{Cx'y'} + x_c \int_V \rho y' \mathrm{d}V + y_c \int_V \rho x' \mathrm{d}V + m x_c y_c.$$

The two integrals on the previous equation are zero. The other products of inertia can be calculated in a similar manner, and the results can be written as follows:

$$\begin{aligned} I_{xy} &= I_{Cx'y'} + m\, x_c\, y_c, \\ I_{xz} &= I_{Cx'z'} + m\, x_c\, z_c, \\ I_{yz} &= I_{Cy'z'} + m\, y_c\, z_c. \end{aligned} \qquad (33)$$

Equations (31) and (33) shows that a translation of axes away from the mass center results in an increase in the moments of inertia. The products of inertia may increase or decrease, depending upon the particular case.

Check Your Progress

4. What is an inertial system?
5. What are the principal axes?

NOTES

7.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Moment of inertia about an axis perpendicular to the plane of an area is known as polar moment of inertia. It may be denoted as I_{zz}. Thus, the moment of inertia about an axis perpendicular to the plane area is called polar moment of inertia and can be written as

   $I_{zz} = \int r^2 dA$

2. Radius of gyration is a mathematical term and is defined by the relation

   $I_{xx} = Ak_x^2 = \int y^2 dA$ or $k_x = \sqrt{\frac{I_{xx}}{A}}$

   Similarly, $k_y = \sqrt{\frac{I_{yy}}{A}}$ and $k_z = \sqrt{\frac{I_{zz}}{A}}$

   where k is known as radius of gyration; I = moment of inertia and A is the cross-sectional area.

3. Mass moment of inertia of a body about an axis is defined as the sum total of product of its elemental masses and square of their distance from the axis. Thus, the mass moment of the body about axis AB is given by

   $I_{AB} = \int r^2 dm$ where r is the distance of element of mass dm from AB.

4. It is a system in the neighborhood of which the gravitational field vanishes, whether this field is produced by matter or by acceleration or rotation of the axes relative to the coordinate system.
5. The directions of the eigenvectors of intertial tensor J are called the principal axes of the rotator relative to the fixed point.

7.7 SUMMARY

- In case of moment of inertia the area is squeezed and kept as a strip of negligible width at a distance k such that there is no change in the moment of inertia.
- The moment of inertia of an area about an axis perpendicular to its plane (polar moment of inertia) at any point is equal to the sum of moments of inertia about any two mutually perpendicular axes through the same point and lying in the plane of the area.

NOTES

- Moment of inertia about any axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroidal axis and the product of area and square of the distance between the two parallel axes.
- The inertia tensor of the rigid body is a symmetric tensor.
- The moment of inertia, also termed as the angular mass or rotational inertia, of a rigid body is a tensor that determines the torque essential for a desired angular acceleration about a rotational axis.
- The moment of inertia about an axis is the square of the reciprocal of the distance from the origin to the momental ellipsoid in the direction of the axis.

7.8 KEY WORDS

- **Polar moment of inertia:** Moment of inertia about an axis perpendicular to the plane of an area is known as polar moment of inertia.
- **Parallel axis theorem:** Moment of inertia about any axis in the plane of an area is equal to the sum of moment of inertia about a parallel centroid axis and the product of area and square of the distance between the two parallel axes. .
- **Mass moment of inertia:** Mass moment of inertia of a body about an axis is defined as the sum total of product of its elemental masses and square of their distance from the axis.
- **Inertial system:** It is a system in the neighbourhood of which the gravitational field vanishes, whether this field is produced by matter or by acceleration or rotation of the axes relative to the coordinate system.

7.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Write a short note on theorems of moments of inertia.
2. Write short notes on the followings:
   a. Polar moment of inertia
   b. Radius of gyration
   c. Mass moment of inertia
   d. Inertial system

Long-Answer Questions

NOTES

1. Give a detailed account of moments and products of inertia.
2. Discuss moment of inertia of a body about any line through the origin of coordinate frame.
3. Explain the momental ellipsoid.
4. Describe rotation of coordinate axes.

7.10 FURTHER READINGS

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UNIT 8 RIGID BODY EQUATIONS OF MOTION

Structure

8.0 INTRODUCTION

In classical mechanics, Euler's rotation equations are a vectorial quasilinear first-order ordinary differential equation describing the rotation of a rigid body, using a rotating reference frame with its axes fixed to the body and parallel to the body's principal axes of inertia. Angular momentum is the rotational equivalent of linear momentum. The total angular momentum of any rigid body can be split into the sum of two main components: the angular momentum of the centre of mass about the origin, plus the spin angular momentum of the object about the centre of mass. In this unit you will discuss principal axes and principal moments about a rigid body. You will understand configuration of a rigid body in space and angular velocity of a rigid body. You will describe angular momentum of a rigid body and Eulerian angles. Compound pendulum is also explained at the end of this unit.

8.1 OBJECTIVES

NOTES

After going through this unit, you will be able to:

- Discuss principal axes and principal moments about a rigid body
- Understand configuration of a rigid body in space and angular velocity of a rigid body
- Describe angular momentum of a rigid body
- Interpret kinetic energy of a rigid body, principal axes and moments of inertia
- Explain Eulerian angles, Euler's equations of motion of a rigid body and torque free motion of a rigid body
- Describe the compound pendulum

8.2 PRINCIPAL AXES AND PRINCIPAL MOMENTS

In classical mechanics, Euler's rotation equations are a vectorial quasilinear first-order ordinary differential equation describing the rotation of a rigid body, using a rotating reference frame with its axes fixed to the body and parallel to the body›s principal axes of inertia.

8.2.1 About a Rigid Body

A body is essentially a system of particles which are distributed in a continuous manner throughout the volume occupied by the body.

If the distance between any pair of particles in a body remains fixed irrespective of its motion in space then the body is said to be a rigid body. This condition is satisfied only approximately by systems that exist in nature. However, in solids, deviations from the above conditions are negligibly small so that a solid can be considered as a rigid body for all practical purposes.

If we assume a rigid body as a system consisting of N discrete particles then the constraints involved are

$$r_{ij} = c_{ij}(i, j = 1, 2, 3, ..., N) \qquad (1)$$

where r_{ij}, which is the distance between the i^{th} and the j^{th} particles, is a constant equal to c_{ij}. However, in a solid, particles are distributed continuously and hence it often becomes necessary to change over from discrete set of particles to continuous distribution of particles. This is done by replacing (i) any summation over the particles by integration over the volume in which the particles are distributed, and (ii) the mass of each particle by infinitesimal mass ρdV of an element of volume dV of the body at which ρ is the density of the material of the body.

8.2.2 Configuration of a Rigid Body in Space: Space Fixed and Body Fixed (or Moving) Coordinate Systems

A rigid body, irrespective of the number of particles of which it is made, has six degrees of freedom. Consequently, the configuration of a rigid body in space can be specified by six generalized coordinates. These coordinates can be chosen in several ways depending upon the exact problem of motion which is considered. To fix up these coordinates it is usual to consider two Cartesian coordinate systems as shown in Figure. 8.1. These are:

(*i*) A frame XYZ called the space-fixed system whose origin and axes are fixed in the space outside the body,

(*ii*) A frame X_1 X_2 X_3 called the body-fixed system whose origin and axes are fixed within the body. It is most convenient to consider the origin of this system to be located at the centre of the mass of the body.

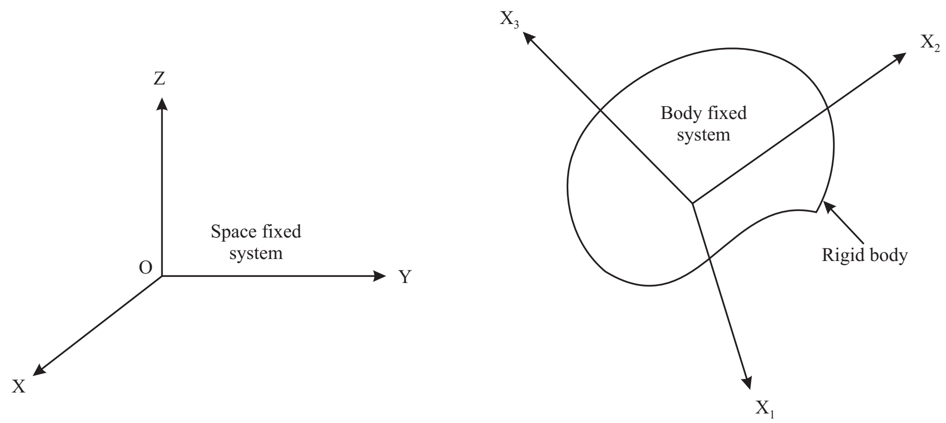

Fig. 8.1 *Space Fixed System and Body Fixed System*

The six generalized coordinates that specify the configuration of the body can then be taken as:

(*i*) Three coordinates of the origin of the body-fixed system with respect to the space-fixed system, and

(*ii*) Three angles which define the orientations of the axes of the body-fixed system relative to the space-fixed system.

It is the body-fixed system that participates in the motion of the body.

8.2.3 Angular Velocity of a Rigid body

Let us refer to the Figure 8.2. Let us consider the origin of the body-fixed system X_1 X_2 X_3 to be the centre of mass of the rigid body under consideration located at the point O.

Let $\vec{R}$ be the position vector of O with respect to origin O_1 of the space-fixed system. Consider any point P of the body whose position vector with respect to O is $\vec{r}$ while that with respect to O_1 is $\vec{\tau}$. Let us now consider an arbitrary infinitesimal displacement of the rigid body which causes a change of $\vec{\tau}$ to $\vec{\tau} + d\vec{\tau}$, where $d\vec{\tau}$ is the vector sum of:

NOTES

(*i*) An infinitesimal translation $d\vec{R}$ of the centre of mass O without any change in the orientation of the body-fixed system with respect to the space-fixed system, and

(*ii*) An infinitesimal translation $\left(d\vec{\phi}\times\vec{r}\right)$ of the centre of mass which results from an infinitesimal rotation of the body through an angle $d\phi$. We may note that an infinitesimal rotation can always be represented by a vector. Here, the magnitude of $d\vec{\phi}$ is $d\phi$ and direction is along the axis of rotation.

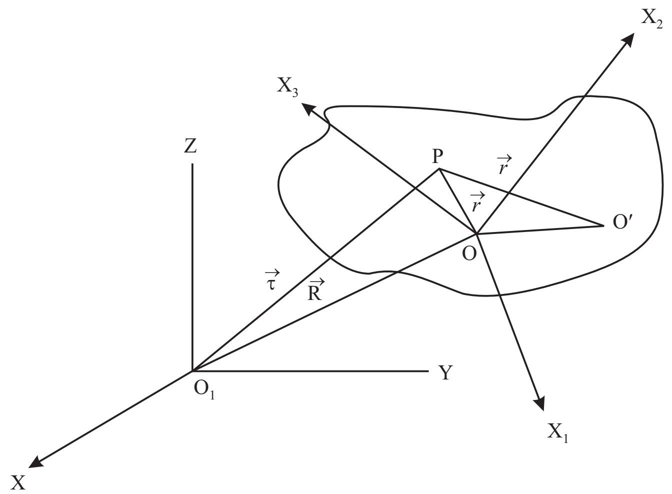

Fig. 8.2 *Angular Velocity*

Thus, we can write

$$d\vec{\tau} = d\vec{R} + d\vec{\phi}\times\vec{r} \tag{2}$$

Let $\vec{v}$ be the instantaneous velocity of the point P as observed from the space-fixed system. We then have

$$\vec{v} = \frac{d\vec{\tau}}{dt} \tag{3}$$

If $\vec{V}$ be the translational velocity of the centre of mass of the body with respect to the space-fixed system, then we have

$$\vec{V} = \frac{d\vec{R}}{dt} \tag{4}$$

NOTES

Differentiating Equation (2) with respect to time, we get

$$\frac{d\vec{\tau}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{\phi}}{dt} \times \vec{r}$$

Using Equation (3) and (4) the above becomes

$$\vec{V} = \vec{V} + \vec{\Omega} \times \vec{r} \tag{5}$$

where $$\vec{\Omega} = \frac{d\vec{\phi}}{dt} \tag{6}$$

is the angular velocity of rotation of the body. The direction of $\vec{\Omega}$ is along the axis of rotation.

Let us now consider the origin of the body-fixed system to be shifted from its centre of mass O to some point O′, such that

$$\overrightarrow{OO'} = \vec{d} \tag{7}$$

Let the position vector of the point P with respect to the new origin be $\vec{r}\,'$

$$\overrightarrow{O'P} = \vec{r}\,' \tag{8}$$

If $\vec{V}\,'$ be the velocity of the new origin O′ as observed from the space-fixed system and if $\vec{\Omega}'$ be the new angular velocity of rotation of the body about an axis passing through O′ and parallel to that passing through the centre of mass O considered above, then we obtain an equation similar to Equation (5) as

$$\vec{v} = \vec{V}\,' + \vec{\Omega}' \times \vec{r}\,' \tag{9}$$

According to Figure 10.2, we have

$$\overrightarrow{OO'} + \overrightarrow{O'P} = \vec{r}$$

Using Equation (7) and (8), we get

$$\vec{r} = \vec{r}\,' + \vec{d} \tag{10}$$

Substituting for $\vec{r}$ given by Equation (10) in Equation (5), we obtain

$$\vec{v} = \vec{V} + \vec{\Omega} \times (\vec{r}\,' + \vec{d})$$

or $$\vec{v} = \vec{V} + \vec{\Omega} \times \vec{r}\,' + \vec{\Omega} + \vec{d} \tag{11}$$

Comparing Equation (9) and (11), we obtain

$$\vec{V}\,' = \vec{v} + \vec{\Omega} \times \vec{d} \tag{12}$$

NOTES

and $$\vec{\Omega}' = \vec{\Omega} \tag{13}$$

From the results given by Equation (12) and (13) we find that

(*i*) the angular velocity of rotation of the rigid body about an axis is independent of the choice of the origin of the body-fixed system provided the direction of the axis of rotation remains the same, and

(*ii*) the velocity of translational motion of the rigid body depends upon the choice of the origin of the body-fixed system.

Check Your Progress

1. What is a rigid body?
2. Does the angular velocity of rotation of the rigid body depend upon the choice of the origin of the body-fixed system?

8.3 ANGULAR MOMENTUM OF A RIGID BODY

Consider a rigid body. Let the body-fixed system have the origin O located at the centre of mass of the body as shown in Figure 8.1. Any general displacement of the body is a translation of the body as a whole and hence of the centre of mass accompanied by a rotation of the body about an axis passing through the origin O of the body-fixed system.

Let us approximate the rigid body by a system of discrete particles 1, 2, ….., N.

Let $\vec{r_i}$ be the instantaneous position vector of the i^{th} particle with respect to the space-fixed system.

If $\vec{p_i}$ be the linear momentum of the i^{th} particle, we get the angular momentum of the i^{th} particle about the origin of the space-fixed system

$$\vec{J_i} = \vec{r_i} \times \vec{p_i} = m_i(\vec{r_i} \times \vec{v_i})$$

where $\vec{v_i}$ is the velocity of the particle as observed from the space-fixed system. Summing over all the particles constituting the body, we get the total angular momentum of the body about the origin of the space-fixed system as

$$\vec{J} = \sum_{i=1}^{N} m_i \left(\vec{r_i} \times \vec{v_i} \right) \tag{14}$$

NOTES

Using Equation (5) we can write the velocity $\vec{v}_i$ in terms of the velocity $\vec{V}$ of the centre of mass and the angular velocity of rotation $\vec{\Omega}$ of the body as

$$\vec{v}_i = \vec{V} + \vec{\Omega} \times \vec{r}_i \tag{15}$$

In view of Equation (15), Equation (14) takes the form

$$\begin{aligned}\vec{J} &= \sum_{i=1}^{N} m_i \cdot \vec{r}_i \times \left(\vec{V} + \vec{\Omega} \times \vec{r}_i\right) \\ &= \sum_{i=1}^{N} \vec{m}_i \left(\vec{r}_i \times \vec{V}\right) + \sum_{i=1}^{N} m_i \cdot \vec{r}_i \times \left(\vec{\Omega} \times \vec{r}_i\right) \end{aligned} \tag{16}$$

By definition, the position vector of the centre of mass of the body with respect to the space-fixed system is

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i} = \frac{\sum m_i \vec{r}_i}{M} \tag{17}$$

where M is the total mass of the body. In view of Equation (17), we may write Equation (16) as

$$\vec{J} = M\left(\vec{R} \times \vec{V}\right) + \sum_{i=1}^{N} m_i \left[\vec{\Omega}\left(\vec{r}_i \cdot \vec{r}_i\right) - \vec{r}_i\left(\vec{r}_i \cdot \vec{\Omega}\right)\right]$$

We have $$\vec{R} \times \vec{V} = \vec{R} \times \frac{d\vec{R}}{dt} = 0$$

and hence, we obtain

$$\vec{J} = \sum_{i=1}^{N} m_i \left[\vec{\Omega} r_i^2 - \vec{r}_i\left(\vec{r}_i \cdot \vec{\Omega}\right)\right]$$

or $$\vec{J} = \sum_{i=1}^{N} m_i \left[r_i^2 \vec{\Omega} - \left(\vec{\Omega} \cdot \vec{r}_i\right)\vec{r}_i\right] \tag{18}$$

8.3.1 Kinetic Energy of a Rigid Body

The kinetic energy T of the rigid body described above is given by

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$

Using the expression for $\vec{v}_i$ given by Equation (15), we get

$$T = \sum \frac{1}{2} m_i \left(\vec{V} + \vec{\Omega} \times \vec{r}_i\right)^2$$

$$= \sum \frac{1}{2} m_i \vec{V} \cdot \vec{V} + \sum \frac{1}{2} m_i \vec{V} . \left(\vec{\Omega} \times \vec{r}_i\right) + \sum \frac{1}{2} m_i \left(\vec{\Omega} \times \vec{r}_i\right) . \left(\vec{\Omega} \times \vec{r}_i\right) \tag{19}$$

we have $$\sum m_i \vec{V}.\left(\vec{\Omega}\times\vec{r_i}\right) = \sum m_i \vec{r_i}\cdot\left(\vec{V}\times\vec{\Omega}\right)$$

$$= \left(\vec{V}\times\vec{\Omega}\right)\cdot\sum m_i \vec{r_i}$$

$$= \left(\vec{V}\times\vec{\Omega}\right)\cdot M\vec{R}$$

or $$\sum m_i \vec{V}\cdot\left(\vec{\Omega}\times\vec{r_i}\right) = 0 \tag{20}$$

Further, we have

$$\left(\vec{\Omega}\times\vec{r_i}\right)\cdot\left(\vec{\Omega}\times\vec{r_i}\right) = \Omega^2 r_1^2 - \left(\vec{\Omega}\cdot\vec{r_i}\right)^2 \tag{21}$$

Using Equation (20) and (21), Equation (19) gives

$$T = \frac{1}{2}MV^2 + \sum\frac{1}{2}m_i\left[\Omega^2 r_i^2 - \left(\vec{\Omega}\cdot\vec{r_1}\right)^2\right] \tag{22}$$

Equation (22) shows that kinetic energy of the rigid body consists of two parts:

(*i*) Translational kinetic energy, given by

$$T_{\text{trans}} = \frac{1}{2}MV^2 \tag{23}$$

as if the total mass M of the body is concentrated at the centre of its mass which moves with the velocity $\vec{V}$.

(*ii*) Rotational kinetic energy, given by

$$T_{\text{rot}} = \sum\frac{1}{2}m_i\left[\Omega^2 r_i^2 - \left(\vec{\Omega}\cdot\vec{r_1}\right)^2\right] \tag{24}$$

We may note that it has been possible to write the total kinetic energy as the sum of the translational and the rotational kinetic energies because of our consideration of the origin of the body-fixed system as the centre of mass of the body.

8.3.2 Principal Axes of Inertia and Principal Moments of Inertia

In the discussion above, for clarity, we used the notations *x, y, z* to denote the components of a position vector. Similarly, we used the notations I_{jk} with $j, k = x, y, z$ to denote the elements of the inertia tensor. Reverting to the notations $X_1\, X_2\, X_3$ for the body fixed (moving) coordinate system, we may write the inertia tensor I as square matrix

$$I = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \tag{25}$$

NOTES

The above symmetrical tensor can be reduced to a diagonal matrix for a particular choice of directions of the axes of the body-fixed system. These particular directions are called **principal axes of inertia**.

The corresponding diagonal elements are called the **principal moments of inertia**.

Thus, the principal axes of inertia are those with respect to which the off-diagonal elements of the inertia tensor vanish.

Let us denote the principal axes by X_1', X_2' and X_3'. Further, denoting $I_{x_1' x_1'} = I_1$, $I_{y_1' y_1'} = I_2$ and $Iz_1,z_1, = I_3$, we may write the inertia tensor as

$$I_d = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \tag{26}$$

In terms of the principal moments of inertia, the rotational kinetic energy given by Equation

$$T_{\text{rot}} = \frac{1}{2}\sum_{j,k} I_{jk}\Omega_j\Omega_k$$

can be written in a simpler form as

$$T_{\text{rot}} = \frac{1}{2}\left(I_1\Omega'^2_1 + I_2\Omega'^2_2 + I_3\Omega'^2_3\right)$$

where Ω_1', Ω_2', Ω_3' are, respectively, the components of angular velocity $\vec{\Omega}$ along the principal axes.

The three components of angular momenta given by Equations

$$J_x = I_{xx}\Omega_x + I_{xy}\Omega_y + I_{xz}\Omega_z$$

$$J_y = I_{yx}\Omega_x + I_{yy}\Omega_y + I_{yz}\Omega_z$$

and $$J_z = I_{zx}\Omega_x + I_{zy}\Omega_y + I_{zz}\Omega_z$$

should read as (considering the body-fixed coordinate system as $X_1 X_2 X_3$)

$$\begin{aligned} J_1 &= I_{11}\Omega_1 + I_{12}\Omega_2 + I_{13}\Omega_3 \\ J_2 &= I_{21}\Omega_1 + I_{22}\Omega_2 + I_{23}\Omega_3 \\ J_3 &= I_{31}\Omega_1 + I_{32}\Omega_2 + I_{33}\Omega_3 \end{aligned} \tag{27}$$

which can be expressed in the form

$$\vec{J} = 1\vec{\Omega} \tag{28}$$

where I is given by the matrix in Equation (25). We may write Equation (28) in matrix form as

$$\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}\begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} \tag{29}$$

If the axes X_1, X_2, X_3 are respectively the principal axes of inertia X_1', X_2', X_3', then Equation (29) reduces to

$$\begin{pmatrix} \hat{j}_1' \\ \hat{j}_2' \\ \hat{j}_3' \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \Omega_1' \\ \Omega_2' \\ \Omega_3' \end{pmatrix} \tag{30}$$

NOTES

and the components of angular momentum $\vec{J}$ along the principal axes are given by

$$\begin{aligned} j_1' &= I_1\Omega_1' \\ j_2' &= I_2\Omega_2' \\ j_3' &= I_3\Omega_3' \end{aligned} \tag{31}$$

or
$$\vec{J} = I_d \vec{\Omega} \tag{32}$$

One important property of principal moments of inertia I_1, I_2 and I_3 is that the sum of any two of these is greater than the remaining third. For example,

$$I_2 + I_3 = \sum m\left(2x_1'^2 + x_2'^2 + x_3'^2\right) > \sum m\left(x_2'^2 + x_3'^2\right)$$

or
$$I_2 + I_3 > I_1$$

8.3.3 Rigid Bodies with Symmetry

In rigid body dynamics it is usual to consider the body to have a definite shape and symmetry. From the symmetry existing in the body, the principal axes become known. For example, in the case of a circular cylinder, the axis of cylindrical symmetry is one of the principal axes.

If the axis of cylindrical symmetry be X_3 then because of symmetry, we obtain

$$I_{13} = -\sum mx_1x_3 = 0, \quad \text{and} \quad I_{23} = -\sum mx_2x_3 = 0,$$

so that $J_3 = I_{33}\Omega_3 = I_3\Omega_3$. Thus, by definition X_3 axis or the axis of cylindrical symmetry is a principal axis. The other two principal axes are in the X_1, X_2 plane with $I_1 = I_2$.

In general, a rigid body is said to be symmetric if two of its principal moments of inertia are equal.

If $I_1 \neq I_2 \neq I_3$, i.e., all the three principal moments of inertia be different, then the rigid body is said to be an **asymmetric top**.

If two principal moments of Inertia be equal, i.e., $I_1 = I_2 \neq I_3$ the rigid body is said to be a **symmetrical top**.

A rigid body is said to be a **spherical top** if all three principal moments of inertia be equal, i.e., $I_1 = I_2 = I_3$. For such a body, the three principal axes may be chosen arbitrarily such that they are mutually perpendicular.

NOTES

Check Your Progress

3. Define the position vector of the centre of mass of the body.
4. What is meant by the kinetic energy of the rigid body?

8.4 EULERIAN ANGLES

As has been discussed above, the configuration of a rigid body which has six degrees of freedom is completely specified by locating the coordinates of a Cartesian system fixed in the body with respect to the coordinate axes of a system fixed in space external to the body.

In other words, it is usual to consider two coordinate systems for describing the motion of a rigid body:

(*i*) A space fixed system *XYZ* whose origin and whose axes are fixed in space, and

(*ii*) A body fixed system $X_1\ X_2\ X_3$ whose origin and axes are fixed within the body so that this system moves along with the body and it is usual to call it the moving system. The above two coordinate frames are shown in the Figure 8.1.

As has been pointed out earlier, it is convenient to choose the origin of the moving system as the centre of mass of the body. Of the six generalized coordinates required to specify the configuration of the body, three are taken as the three Cartesian coordinates of the centre of mass of the body, i.e., the origin of the moving system with respect to the space-fixed system. About the remaining three let us look into the following:

Let

$\hat{i}, \hat{j}$ and $\hat{k}$ be, respectively, the unit vectors along the *X*-, *Y*- and *Z*-axes.

$\hat{i}', \hat{j}'$ and $\hat{k}'$ be respectively the unit vectors along the X_1-, X_2- and X_3-axes.

α_1, β_1, and γ_1 be the direction cosines of the *X*-axis relative to the X_1-, X_2- and X_3-axes, respectively.

α_2, β_2, and γ_2 be the direction cosines of the *Y*-axis relative to X_1-, X_2- and X_3-axes, respectively.

α_3, β_3 and γ_3 be the direction cosines of the *Z*-axis relative to X_1-, X_2- and X_3-axes, respectively.

We then have the relations

$$\hat{i}' = \alpha_1\hat{i} + \beta_1\hat{j} + \gamma_1\hat{k}$$

$$\hat{j}' = \alpha_2\hat{i} + \beta_2\hat{j} + \gamma_2\hat{k} \quad (33)$$

$$\hat{k}' = \alpha_3\hat{i} + \beta_3\hat{j} + \gamma_3\hat{k}$$

NOTES

We further have

$$\begin{aligned} \hat{i}'\cdot\hat{i}' &= \alpha_1^2 + \beta_1^2 + \gamma_1^2 = 1 \\ \hat{j}'\cdot\hat{j}' &= \alpha_2^2 + \beta_2^2 + \gamma_2^2 = 1 \\ \hat{k}'\cdot\hat{k}' &= \alpha_3^2 + \beta_3^2 + \gamma_3^2 = 1 \end{aligned} \quad (34)$$

and

$$\begin{aligned} \hat{i}'\cdot\hat{j}' &= \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = 0 \\ \hat{j}'\cdot\hat{k}' &= \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 = 0 \\ \hat{k}'\cdot\hat{i}' &= \alpha_3\alpha_1 + \beta_3\beta_1 + \gamma_3\gamma_1 = 0 \end{aligned} \quad (35)$$

We find nine direction cosines connected by six relations. Thus, three remain unconnected. However, these three are not independent of each other and as such they cannot be taken as the remaining three generalized coordinates for the specification of the configuration of the rigid body.

Various sets of the remaining three generalized coordinates have been proposed. The most common and useful of them are the Eulerian angles. They refer to the angles corresponding to three successive rotations of the space-fixed system performed in a particular sequence or order, such that at the end, the axes of the space-fixed system coincide with those of the body-fixed system. Clearly, the Eulerian angles give the orientations of the axes of the body-fixed system relative to the space-fixed system.

In the following, we consider the three successive rotations of the space-fixed system to define Eulerian angles.

First Rotation

The space-fixed system (*XYZ*) is rotated about the *Z*-axis counter-clockwise by an angle ϕ, such that the *X*- and *Y*-axes, respectively, take the new positions X' and $Y'$and the new *Y*–*Z* plane, namely the Y'–Z' plane contains the axis X_3 of the body-fixed system as shown in the Figure 8.3.

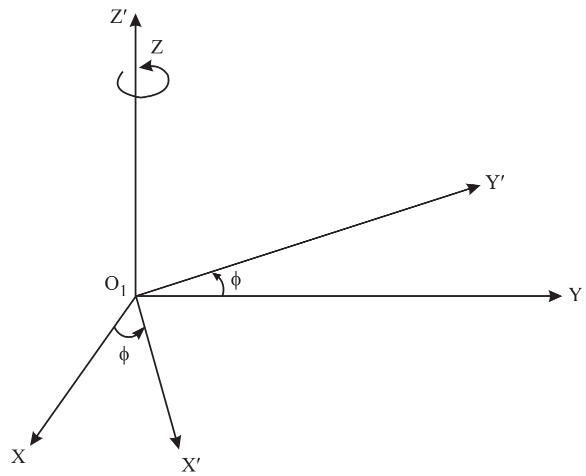

Fig. 8.3 *First Rotation*

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Let $\hat{i}_1', \hat{j}_1'$ and $\hat{k}_1'$ be, respectively, the unit vectors along the transformed set of axes X', Y' and Z'.

We then have

$$\begin{aligned}\hat{i}_1' &= \hat{i}\cos\phi + \hat{j}\sin\phi \\ \hat{j}_1' &= -\hat{i}\sin\phi + \hat{j}\cos\phi \\ \hat{k}_1' &= \hat{k}\end{aligned} \tag{36}$$

The above equations can be written in the matrix form as

$$\begin{pmatrix}\hat{i}_1' \\ \hat{j}_1' \\ \hat{k}_1'\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{i} \\ \hat{j} \\ \hat{k}\end{pmatrix} \tag{37}$$

The matrix of transformation from XYZ to $X'\,Y'Z'$ is thus,

$$D = \begin{pmatrix}\cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1\end{pmatrix} \tag{38}$$

Second Rotation

The transformed system $X'Y'Z'$ is rotated about the X'-axis counter-clockwise by an angle θ, such that Z'-axis which is the same as the Z-axis coincides with the axis X_3 of the body-fixed system and the transformed X'– Y' plane becomes the X_1–X_2 plane of the body-fixed system as shown in Figure 8.4.

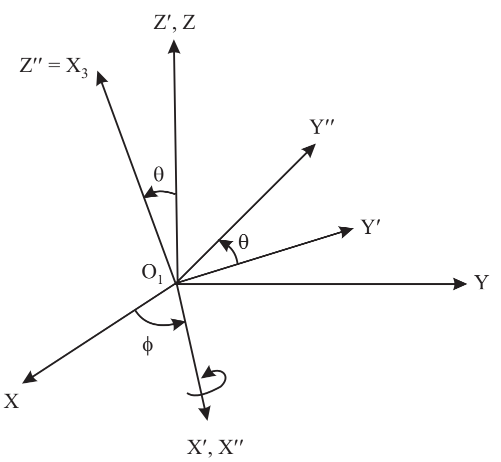

Fig. 8.4 *Second Rotation*

Let $\hat{i}_1'', \hat{j}_1'', \hat{k}_1''$ be, respectively, the unit vectors along the transformed set of axes X'', Y'' and Z''. We then have

$$\hat{i}_1'' = \hat{i}_1'$$

$$\hat{j}_1'' = \hat{j}_1' \cos\theta + \hat{k}_1' \sin\theta \tag{39}$$
$$\hat{k}_1'' = -\hat{j}_1' \sin\theta + \hat{k}_1' \cos\theta$$

In matrix form, the above equations can be written as

$$\begin{pmatrix} \hat{i}_1'' \\ \hat{j}_1'' \\ \hat{k}_1'' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{i}_1' \\ \hat{j}_1' \\ \hat{k}_1' \end{pmatrix} \tag{40}$$

The matrix of transformation from the $X'Y'Z'$ system to $(X''Y''Z'')$ system is thus

$$C = \begin{pmatrix} 1 & 0 & 0 \\ \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{pmatrix} \tag{41}$$

NOTES

Third Rotation

The new system $(X''Y''Z'')$ obtained after the second rotation is rotated about the Z'' $(= X_3)$ axis counter-clockwise by an angle ψ such that the transformed axis X''' coincides with X_1-axis while the transformed axis Y''' coincides with X_2-axis of the body-fixed system as illustrated in Figure 8.5.

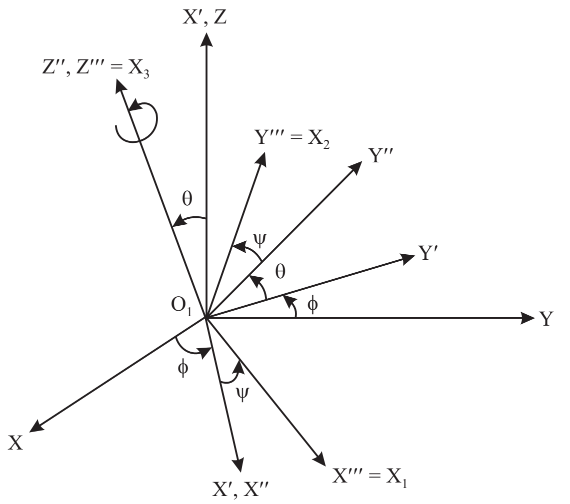

Fig. 8.5 *Third Rotation*

Let $\hat{i}_1'''$, $\hat{j}_1'''$, $\hat{k}_1'''$ be, respectively, the unit vectors along the transformed axes X''', Y''' and Z'''. We get according to the operation performed

$$\hat{i}_1''' = \hat{i}_1'' = \hat{i}_1'' \cos\psi + \hat{j}_1'' \sin\psi$$
$$\hat{j}_1''' = \hat{j}_1''' = \hat{j}_1'' = -\hat{i}_1'' \sin\psi + \hat{j}_1'' \cos\psi \tag{42}$$
$$\hat{k}_1''' = \hat{k}' = \hat{k}_1''$$

In matrix form we may write the above equations as

$$\begin{pmatrix} \hat{i}_1''' \\ \hat{j}_1''' \\ \hat{k}_1''' \end{pmatrix} = \begin{pmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i}_1'' \\ \hat{j}_1'' \\ \hat{k}_1'' \end{pmatrix} \tag{43}$$

The matrix of transformation from $(X''Y''Z'')$ system to $X'''Y'''Z'''$ (or $X_1 X_2 X_3$) system is thus

$$B = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{44}$$

The complete transformation from (XYZ) system to $(X_1 X_2 X_3)$ system is thus in matrix form given by

$$\begin{pmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{pmatrix} = B \begin{pmatrix} \hat{i}_1'' \\ \hat{j}_1'' \\ \hat{k}_1'' \end{pmatrix} = BC \begin{pmatrix} \hat{i}_1'' \\ \hat{j}_1'' \\ \hat{k}_1'' \end{pmatrix} = BCD \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

or

$$\begin{pmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{pmatrix} = A \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} \tag{45}$$

where $A = BCD$
(46)

Using the matrices for B given by Equation (44), C given by Equation (41) and D given by Equation (38), we obtain

$$\text{A}=\begin{pmatrix} \cos\psi\cos\phi-\cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi+\cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi-\cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi+\cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \tag{47}$$

Since all the elements of the matrix A are real, the matrix A itself is real.

Each of the matrices B, C and D corresponds to orthogonal transformation because the transformation of the axes is caused by simple rotations. We, thus, have

$$\widetilde{D} = D^{-1},\ \widetilde{C} = C^{-1},\ \widetilde{B} = B^{-1} \tag{48}$$

We have the total transformation matrix $A = BCD$

Taking transpose of the above, we get

$$\widetilde{A} = \widetilde{BCD} = \widetilde{D}\widetilde{C}\widetilde{B} \tag{49}$$

or

$$\widetilde{A} = D^{-1}C^{-1}B^{-1} = (BCD)^{-1} = A^{-1} \tag{50}$$

Thus, A is also orthogonal.

8.4.1 Equations of Motion of a Rigid Body: Euler's Equations

There exist several methods to analyse the dynamics of a rigid body. One such method is due to Euler in which the analysis is made in terms of the body-fixed frame of reference or the moving coordinate system which rotates with the body. Simplification arises because relative to this frame, the moments of inertia and products of inertia are time-independent while relative to space-fixed system they are functions of time.

8.4.2 Euler's Equation for Force Free Motion

We know that a rigid body undergoes pure rotational motion about an axis passing through a fixed point in the body when a net external torque about that axis acts on the body. The external torque $\vec{\Gamma}$ and the angular momentum $\vec{J}$ of the body about the axis of rotation are related according to

$$\vec{\Gamma} = \frac{d\vec{J}}{dt} \tag{51}$$

where the time derivative of $\vec{J}$ is calculated relative to the space-fixed system external to the body.

If $\vec{\Omega}$ be the angular velocity of rotation of the body then the time derivatives relative to space-fixed system of axes and body-fixed system of axes are given by

$$\left(\frac{d}{dt}\right)_{\text{space}} = \left(\frac{d}{dt}\right)_{\text{body}} + \vec{\Omega} X \tag{52}$$

In view of the relation expressed by Equation (52), we may write Equation (51) as

$$\vec{\Gamma} = \left(\frac{d\vec{J}}{dt}\right)_{\text{body}} + \vec{\Omega}\times\vec{J} \tag{53}$$

If, for convenience, we choose the axes of the body-fixed system as the principal axes of the body, we get

$$\vec{J} = \hat{i}I_1\Omega_1 + \hat{j}I_2\Omega_2 + \hat{k}I_3\Omega_3 \tag{54}$$

where Ω_1, Ω_2, Ω_3 are, respectively, the components of the angular velocity $\vec{\Omega}$ along the principal axes along which the unit vectors are $\hat{i}, \hat{j}, \hat{k}$, while I_1, I_2, I_3 are the principal moments of inertia.

Since, relative to the body-fixed system, the principal moments of inertia and the unit vectors are stationary or time-independent, we get from Equation (54)

$$\left(\frac{d\vec{J}}{dt}\right)_{\text{body}} = \hat{i}I_1\dot{\Omega}_1 + \hat{j}I_2\dot{\Omega}_2 + \hat{k}I_3\dot{\Omega}_3 \quad (55)$$

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The component of $\vec{\Gamma}$ along the principal axis along which the unit vector is $\hat{i}$, is given by

$$\vec{\Gamma} = \hat{i}\cdot\vec{\Gamma} = \hat{i}\cdot\left[\left(\frac{d\vec{J}}{dt}\right)_{\text{body}} + \vec{\Omega}\times\vec{J}\right] \quad \text{[usinig Equation (53)]}$$

$$= \hat{i}\cdot\left(\frac{d\vec{J}}{dt}\right)_{\text{body}} + \hat{i}\left(\vec{\Omega}\times\vec{J}\right)$$

$$= I_1\dot{\Omega}_1 + \hat{i}\cdot\left[\hat{i}\left(\Omega_2 j_3 - \Omega_3 j_2\right) + \hat{j}\left(\Omega_3 j_1 - \Omega_1 j_3\right) + \hat{k}\left(\Omega_1 j_2 - \Omega_2 j_1\right)\right]$$

or $$\Gamma_1 = I_1\dot{\Omega}_1 + \left(\Omega_2 j_3 - \Omega_3 j_2\right)$$

or $$\Gamma_1 = I_1\dot{\Omega}_1 + \Omega_2 I_3\Omega_3 - \Omega_3 I_2\Omega_2 \quad \text{[using Equation (54)]}$$

or $$\Gamma_1 = I_1\dot{\Omega}_1 - \left(I_2 - I_3\right)\Omega_2\Omega_3 \quad (56)$$

Similarly, we obtain for the other two components of the torque $\vec{\Gamma}$ along the remaining principal axes as

$$\Gamma_2 = I_2\dot{\Omega}_2 - \left(I_3 - I_1\right)\Omega_3\Omega_1 \quad (57)$$
$$\Gamma_3 = I_3\dot{\Omega}_3 - \left(I_1 - I_2\right)\Omega_1\Omega_2 \quad (58)$$

Restricting our considerations to force-free motion of the rigid body, we get

(*i*) Potential energy of the body = 0

(*ii*) Kinetic energy of the body = Rotational kinetic energy (T_{rot})

so that the Lagrangian L of the body becomes

$$L = T_{\text{rot}}$$

Choosing the axes of the body-fixed or the moving system as the principal axes of the body, we get the Lagrangian as

$$L = \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right) \quad (59)$$

Further, for convenience, let us choose the generalized coordinates corresponding to the three rotational degrees of freedom as the Eulerian angels ψ, θ and ϕ. We may then write the Lagrangian as

$$L = L\left(\psi, \theta, \phi, \dot{\psi}, \dot{\theta}, \dot{\phi}\right) \quad (60)$$

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The Lagrange's equation for the coordinate ψ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\psi}}\right) = \frac{\partial L}{\partial \psi}$$

$$\text{or } \frac{d}{dt}\sum_{i=1}^{3}\frac{\partial L}{\partial \Omega_i}\frac{\partial \Omega_i}{\partial \dot{\psi}} - \sum_{i=1}^{3}\frac{\partial L}{\partial \Omega_i}\frac{\partial \Omega_i}{\partial \psi} = 0 \qquad (61)$$

The components of the angular velocity $\vec{\Omega}$ can be expressed in terms of the Eulerian angels as

$$\begin{aligned}
\Omega_1 &= \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\
\Omega_2 &= \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \qquad (62) \\
\Omega_3 &= \dot{\phi}\cos\theta + \dot{\psi}
\end{aligned}$$

From the above we obtain

$$\begin{aligned}
\frac{\partial \Omega_1}{\partial \psi} &= \phi\sin\theta\cos\psi - \dot{\theta}\sin\psi = \Omega_2 \\
\frac{\partial \Omega_2}{\partial \psi} &= -\phi\sin\theta\sin\psi - \dot{\theta}\cos\psi = -\Omega_1 \qquad (63) \\
\frac{\partial \Omega_3}{\partial \psi} &= 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \Omega_1}{\partial \dot{\psi}} &= 0 \\
\frac{\partial \Omega_2}{\partial \dot{\psi}} &= 0 \qquad (64) \\
\frac{\partial \Omega_3}{\partial \dot{\psi}} &= 1
\end{aligned}$$

From Equation (59), we get

$$\begin{aligned}
\frac{\partial L}{\partial \Omega_1} &= I_1\Omega_1 \\
\frac{\partial L}{\partial \Omega_2} &= I_2\Omega_2 \qquad (65) \\
\frac{\partial L}{\partial \Omega_3} &= I_3\Omega_3
\end{aligned}$$

8.4.3 Euler's Equation in a Force Field

Since a rigid body, in general, has six degrees of freedom, its motion can be described in terms of six independent coordinates. Thus, the general equations of motion of a rigid body are six in number. In the Newtonian formulation, three of these are given by

$$\dot{\vec{p}} = \sum \dot{\vec{p}} = \sum \vec{f} = \vec{F}$$

where $\dot{\vec{p}}$ is the linear momentum of any particle, and $\vec{f}$ is the force acting on the particle.

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The summation is carried over all the particles of the body. Here, the total force $\vec{F}$ includes, in principle, both external as well as internal forces.

We may start with the fundamental equations expressed as

$$\left(\frac{d\vec{P}}{dt}\right)_{fixed} = \vec{F} \text{ and } \left(\frac{d\vec{M}}{dt}\right)_{fixed} = \vec{k}$$

The designation 'fixed' is written explicitly since the above relations (from Newtonian mechanics) are valid only in an inertial frame of reference. The moving system X_1, X_2, X_3, is fixed in the rigid body and hence rotates with angular velocity $\vec{\Omega}$. With reference to figure 10.3, if the radius vector of a point in the system changes from $\vec{r}$ to $\vec{r} + \overrightarrow{\delta r}$ then the geometrical situation is correctly represented, if we write

$$\overrightarrow{\delta r} = \overrightarrow{\delta\theta} \times \vec{r}$$

where $\overrightarrow{\delta\theta}$ is a vector whose magnitude is equal to the infinitesimal rotation angle $\delta\theta$ and having direction along the instantaneous axis of rotation. Similarly, we can write

$$\overrightarrow{\delta v} = \overrightarrow{\delta\theta} \times \vec{v}$$

Let us consider an arbitrary vector $\vec{A}$. The change in this vector in time *dt* with respect to the fixed axis differs from the corresponding change with respect to the axis moving with the rigid body, only by the effects of the rotation of the body axes. In other words, we may write

$$\left(d\vec{A}\right)_{fixed} = \left(d\vec{A}\right)_{moving} + \left(d\vec{A}\right)_{rot}$$

However, the change in the components of a vector, arising solely from an infinitesimal rotation $d\theta$ of the coordinate axes is exactly given by Equation (91). Hence, we get

$$\left(d\vec{A}\right)_{fixed} = \left(d\vec{A}\right)_{moving} + d\theta \times \vec{A}$$

The time rate of change of $\vec{A}$ is then

$$\left(\frac{d\vec{A}}{dt}\right)_{fixed} = \left(\frac{d\vec{A}}{dt}\right)_{moving} + \vec{\Omega} \times \vec{A}$$

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The six equations of motion with respect to the body-fixed system are thus

$$\vec{F} = \left(\frac{d\vec{P}}{dt}\right)_{moving} + \vec{\Omega}\times\vec{P}$$

$$\vec{K} = \left(\frac{d\vec{M}}{dt}\right)_{moving} + \vec{\Omega}\times\vec{M}$$

Choosing the axes of the moving system to coincide with the principal axes of the body and taking $P = M\vec{V}$ and $M_i = I_i\Omega_i$, etc., we obtain

$$F_1 = M\left(\frac{dV_1}{dt} + \Omega_2 V_3 - \Omega_3 V_2\right)$$

$$F_2 = M\left(\frac{dV_2}{dt} + \Omega_3 V_1 - \Omega_1 V_3\right)$$

$$F_3 = M\left(\frac{dV_3}{dt} + \Omega_1 V_2 - \Omega_2 V_1\right)$$

$$K_1 = I_1\frac{d\Omega_1}{dt}\Omega_2\Omega_3(I_3 - I_2)$$

$$K_2 = I_2\frac{d\Omega_2}{dt}\Omega_3\Omega_1(I_1 - I_3)$$

$$K_3 = I_3\frac{d\Omega_3}{dt}\Omega_1\Omega_2\ I_2 - I_1)$$

Substituting the above results [Equation (62) to (85)] in Equation (61), we obtain

$$\frac{d}{dt}(I_3\Omega_3) - [I_1\Omega_1\Omega_2 - I_2\Omega_2\Omega_1] = 0$$

or $$(I_1 - I_2)\Omega_1\Omega_2 - I_3\Omega_3$$

or $$I_3\dot{\Omega}_3 = (I_1 - I_2)\Omega_1\Omega_2 \qquad (66)$$

Similarly, we obtain the equations for $\dot{\Omega}_1$ and $\dot{\Omega}_2$ as

$$I_1\dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3 \qquad (67)$$

$$I_2\dot{\Omega}_2 = (I_3 - I_1)\Omega_3\Omega_1 \qquad (68)$$

$$\dot{\vec{p}} = \sum\dot{\vec{p}} = \sum\vec{f} = \vec{F} \qquad (69)$$

$$\frac{\overrightarrow{dp}}{dt} = \vec{F} \text{ and } \left(\frac{d\vec{M}}{dt}\right)_{fixed} = \vec{K} \qquad (70)$$

$$\delta\vec{r} = \delta\vec{\theta}\times\vec{r} \qquad (71)$$

$$\delta\vec{v} = \delta\vec{\theta}\times\vec{v} \qquad (72)$$

NOTES

$$\left(d\vec{A}\right)_{fixed} = \left(d\vec{A}\right)_{moving} + \left(d\vec{A}\right)_{rotational} \tag{73}$$

$$\left(\frac{d\vec{A}}{dt}\right)_{fixed} = \left(\frac{d\vec{A}}{dt}\right)_{moving} + \vec{\Omega}\times\vec{A} \tag{74}$$

$$\vec{F} = \left(\frac{d\vec{P}}{dt}\right)_{moving} + \vec{\Omega}\times\vec{P} \tag{75}$$

$$\vec{K} = \left(\frac{d\vec{M}}{dt}\right)_{moving} + \vec{\Omega}\times\vec{M} \tag{76}$$

Choosing the axes of the moving system to coincide with the principal axes of the body and taking $P_i = M\vec{V}_i$ and $M_i = I_i\Omega_i$, $M_i = I_i\Omega_i$, etc., we obtain

$$\begin{aligned} F_1 &= M\left(\frac{dV_1}{dt} + \Omega_2 V_3 - \Omega_3 V_2\right) \\ F_2 &= M\left(\frac{dV_2}{dt} + \Omega_3 V_1 - \Omega_1 V_3\right) \\ F_3 &= M\left(\frac{dV_3}{dt} + \Omega_1 V_2 - \Omega_2 V_1\right) \end{aligned} \tag{77}$$

$$\begin{aligned} K_1 &= I_1\frac{d\Omega_1}{dt}\Omega_2\Omega_3\left(I_3 - I_2\right) \\ K_2 &= I_2\frac{d\Omega_2}{dt}\Omega_3\Omega_1\left(I_1 - I_3\right) \\ K_3 &= I_3\frac{d\Omega_3}{dt}\Omega_1\Omega_2\left(I_2 - I_1\right) \end{aligned} \tag{78}$$

Equations (77) and (78) are Euler's equations for the motion of a rigid body in a force field.

We may note that Equation (66) for Ω_3 is the Lagrange's equation for the coordinate ψ but Equation (67) and (68) are not Lagrange's equations for the coordinates θ and ϕ.

8.4.4 Torque Free Motion of a Rigid Body

Euler's equations obtained in the previous sections can be conveniently applied to describe the motion of the rigid body when no net force or no net torque acts on the body. We first consider the torque free motion.

Consider a rigid body rotating about an axis passing through the centre of mass of the body. Let us choose the centre of mass, which is a fixed point within the body, as the origin of the principal axes of the body. Considering

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no torque to be acting on the body, Euler's equations given by Equation (56), (57) and (58) reduce respectively to

$$I_1 \dot{\Omega}_1 = (I_2 - I_3)\, \Omega_2\, \Omega_3 \qquad (79)$$

$$I_2 \dot{\Omega}_2 = (I_3 - I_1)\, \Omega_3\, \Omega_1 \qquad (80)$$

$$I_3 \dot{\Omega}_3 = (I_1 - I_2)\, \Omega_1\, \Omega_2 \qquad (81)$$

Multiplying Equation (79), (80) and (81) respectively by Ω_1, Ω_2 and Ω_3 and adding, we get

$$I_1 \dot{\Omega}_1 \Omega_1 + I_2 \dot{\Omega}_2 \Omega_2 + I_3 \dot{\Omega}_3 \Omega_3 = [I_2 - I_3 + I_3 - I_1 + I_1 - I_2]\, \Omega_1\, \Omega_2\, \Omega_3 = 0$$

or $$\frac{d}{dt}\left(\frac{1}{2} I_1 \Omega_1^2 + \frac{1}{2} I_2 \Omega_2^2 + \frac{1}{2} I_3 \Omega_3^2\right) = 0$$

or $$\frac{1}{2} I_1 \Omega_1^2 + \frac{1}{2} I_2 \Omega_2^2 + \frac{1}{2} I_3 \Omega_3^2 = \text{a constant}$$

Thus, $$T_{\text{rot}} = \frac{1}{2}\left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2\right) = \text{a constant} \qquad (82)$$

Equation (82) shows that the kinetic energy of rotation of the body is an integral of motion.

Since no torque acts on the body, the total angular momentum $\vec{J}$ of the rotating body is another integral of motion $\left(\frac{d\vec{J}}{dt} = \text{torque} = 0 \text{ or } \vec{J} = \text{a constant}\right)$. Thus, we have

$$\vec{J} = \hat{i}\, I_1 \Omega_1 + \hat{j}\, I_2 \Omega_2 + \hat{k}\, I_3 \Omega_3 = \text{a constant} \qquad (83)$$

We have

$$\vec{\Omega} \cdot \vec{J} = \left[\hat{i}\Omega_1 + \hat{j}\Omega_2 + \hat{k}\Omega_3\right] \cdot \left[\hat{i} I_1 \Omega_1 + \hat{j} I_2 \Omega_2 + \hat{k}\, I_3 \Omega_3\right]$$

$$= I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2 \qquad (84)$$

Combining Equation (82) and (84) we obtain

$$2T_{\text{rot}} = \vec{\Omega} \cdot \vec{J} = \text{a constant} \qquad (85)$$

(i) Inertia Ellipsoid

We may note that the motion of a rigid body depends on the structure of the body through the quantities (numbers) I_1, I_2 and I_3. Hence, any two bodies which have the same principal moments of inertia move in exactly the same manner although they may have different shapes. The simplest geometrical shape for a body having three given principal moments is that of a homogeneous ellipsoid. Hence, it often becomes convenient to describe the motion of a rigid body in terms of the motion of equivalent ellipsoid. Such a description of a rigid body was due to **Poinsot** which has the advantage of

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providing a geometrical description of the motion without trying to obtain a complete solution of the problem.

Poinsot's construction can be understood as explained below. The kinetic energy of the rotating rigid body relative to a coordinate system whose axes are the principal axes is given by

$$T_{\text{rot}} = \frac{1}{2}\vec{\Omega}\cdot\vec{J} = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)$$

We may write

$$2T = I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2 = I\Omega^2 \qquad (86)$$

where I is the moment of inertia of the body about the axis of rotation.

Let $\hat{n}$ be a unit vector in the direction of $\vec{\Omega}$, so that

$$\vec{\Omega} = \Omega\,\hat{n} \qquad (87)$$

Let the direction cosines of the axis of rotation be α, β and γ.

We then get

$$\hat{n} = \alpha i + \beta\hat{j} + \gamma\hat{k} \qquad (88)$$

and we can write the moment of inertia of the body about this axis as

$$I = \alpha^2 I_{xx} + \beta^2 I_{yy} + \gamma^2 I_{zz} + 2I_{xy}\alpha\beta + 2I_{yz}\beta\gamma + 2I_{zx}\gamma\alpha \qquad (89)$$

Let us now define a vector $\vec{P}$ according to

$$\vec{P} = \frac{\hat{n}}{\sqrt{I}} \qquad (90)$$

Using Equation (87) the above becomes

$$\vec{P} = \frac{\vec{\Omega}}{\Omega\sqrt{I}} = \frac{\vec{\Omega}}{\sqrt{I\Omega^2}}$$

In view of Equation (86) the above can be written as

$$\vec{P} = \frac{\vec{\Omega}}{\sqrt{2T}} = \frac{1}{\sqrt{2T}}(\hat{i}\Omega_1 + \hat{j}\Omega_2 + \hat{k}\Omega_3) \qquad (91)$$

Further, we may write $\vec{P}$ in terms of its components P_1, P_2, P_3 as

$$\vec{P} = \hat{i}P_1 + \hat{j}P_2 + \hat{k}P_3 \qquad (92)$$

Comparing Equation (91) and (92) we obtain

$$P_1 = \frac{\Omega_1}{\sqrt{2T}}, \quad P_2 = \frac{\Omega_2}{\sqrt{2T}}, \quad P_3 = \frac{\Omega_3}{\sqrt{2T}} \qquad (93)$$

In view of Equation (93), Equation (86) gives

$$I_1P_1^2 + I_2P_2^2 + I_3P_3^2 = 1 \qquad (94)$$

Equation (94) is the equation of an ellipsoid and is called the equation of inertia ellipsoid.

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(ii) Invariable Plane

Consider a rigid body rotating about a fixed point, say *O,* without the action of any external force or torque. The angular momentum vector $\vec{J}$ is a constant of motion and has a fixed direction in space as shown in Figure 8.6. The line along the fixed direction of $\vec{J}$ is called the invariable line. We have for force/torque free motion

$$\vec{\Omega}\cdot\vec{J} = 2T = \text{constant}$$

Clearly, the projection of $\vec{\Omega}$ along $\vec{J}$ is $\Omega \cos\theta$ which is constant and hence the tip of $\vec{\Omega}$ describes a plane called the invariable plane. To an observer fixed in the body-fixed coordinate system the angular velocity vector $\vec{\Omega}$ is found to precess about the angular momentum vector $\vec{J}$.

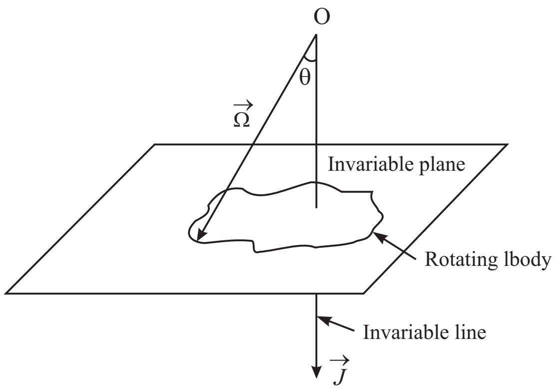

Fig. 8.6 *Fixed Direction*

For force-free motion of the rigid body we have

$$\vec{P}\cdot\vec{J} = \frac{\vec{\Omega}\cdot\vec{J}}{\sqrt{2T}} = \sqrt{2T} = \text{constant} \tag{95}$$

The above shows that the tip of the vector $\vec{P}$ also describes an invariable plane. It can be seen that this invariable plane is the tangent plane at the point *P* of the inertia ellipsoid.

The distance between the origin of the ellipsoid and the tangent plane at the point *P* is

$$d = P\cos\theta = \frac{\vec{P}.\vec{J}}{J} = \frac{\vec{\Omega}\cdot\vec{J}}{J\sqrt{2T}} = \frac{\sqrt{2T}}{J} = \text{constant} \tag{96}$$

NOTES

As a consequence we find that as the angular velocity vector $\vec{\Omega}$ and hence $\vec{P}$ changes with time, the inertia ellipsoid rolls on the invariable plane with the centre of the ellipsoid at a constant height above the plane.

The curve traced out on the invariable plane by the point of contact with the ellipsoid is called **herpolhode** and the corresponding curve described on the ellipsoid is called **polhode**. We find that the polhode undergoes pure rolling on the herpolhode in the invariable plane (Refer to Figure 8.7).

In the case of a symmetrical rigid body rotating about the symmetry axis (z-axis) we have $I_1 = I_2$ and we find the inertia ellipsoid to be an ellipsoid of revolution. Vector $\vec{P}$ and hence vector $\vec{\Omega}$ remains constant in magnitude. As a result, the polhode becomes a circle about the symmetry axis of the ellipsoid and herpolhode is a circle on the invariable plane. The angular velocity vector $\vec{\Omega}$ describes a cone called the **body cone**. As observed by the observer in the space-fixed system $\vec{\Omega}$ moves also on the surface of a cone called the space cone (Refer to Figure 8.8).

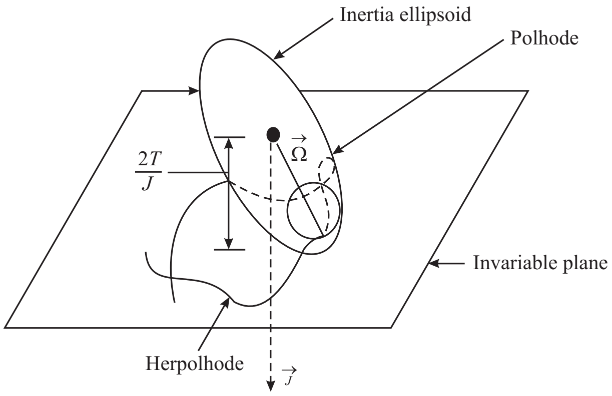

Fig. 8.7 *Invariable Plane*

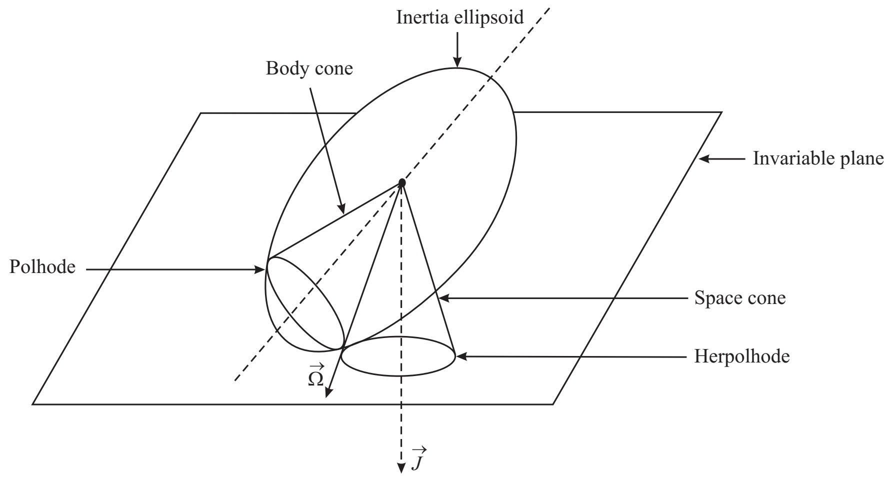

Fig. 8.8 *Space Cone*

8.4.5 Force Free Motion of a Symmetrical Rigid Body

As another application, we use in the following, Euler's equations to discuss force free motion of a symmetrical rigid body.

Choosing the symmetry axis as the principal z-axis we get $I_1 = I_2$ and Equation (56), (57) and (58) give

$$I_1 \dot{\Omega}_1 = (I_1 - I_3)\, \Omega_2\, \Omega_3 \tag{97}$$

$$I_1 \dot{\Omega}_2 = (I_3 - I_1)\, \Omega_3\, \Omega_1 = -(I_1 - I_3)\, \Omega_3\, \Omega_1 \tag{98}$$

$$I_3 \dot{\Omega}_3 = 0 \tag{99}$$

Equation (99) yields

$$\Omega_3 = \text{constant} \tag{100}$$

i.e., the component of angular velocity along the symmetry axis is a constant.

Putting
$$\Omega_0 = \frac{I_1 - I_3}{I_1}\Omega_3 \text{ (which is a constant)} \tag{101}$$

we may write Equation (97) and (98) as

$$\dot{\Omega}_1 = \Omega_0\, \Omega_2 \tag{102}$$

$$\dot{\Omega}_2 = -\, \Omega_0\, \Omega_1 \tag{103}$$

Differentiating Equation (102) with respect to time we get

$$\ddot{\Omega}_1 = \Omega_0\, \dot{\Omega}_2$$

Substituting Equation (103) in the above we obtain

$$\ddot{\Omega}_1 = -\, \Omega_0^2\, \Omega_1$$

or
$$\ddot{\Omega}_1 + \Omega_0^2\, \Omega_1 = 0 \tag{104}$$

Solution of Equation (104) can be put in the form

$$\Omega_1 = A \sin\, \Omega_0 t \tag{105}$$

where A is some constant. We have chosen the phase constant such that at $t = 0,\ \Omega_1 = 0$.

From Equation (105) we get

$$\dot{\Omega}_1 = A\Omega_0 \cos \Omega t \tag{106}$$

Using the above in Equation (102) we obtain

$$A\Omega_0 \cos\Omega_0 t = \Omega_0\, \Omega_2$$

or
$$\Omega_2 = A \cos \Omega_0 t \tag{107}$$

Combining Equation (105) and (107) we obtain

$$\Omega_1^2 + \Omega_2^2 = A^2 \tag{108}$$

which is the equation of a circle of radius A. Let us now consider the vector $\vec{\Omega}_P$ in the x-y plane as

$$\vec{\Omega}_P \;=\; \hat{i}\Omega_1 + \hat{j}\Omega_2 = \hat{i}\,A\sin\Omega t + \hat{j}\,A\cos\Omega t$$

The above gives

$$\vec{\Omega}_P\,| = \;A = \text{constant} \tag{109}$$

We further find that the vector $\vec{\Omega}_P$ rotates about the symmetry axis with constant angular frequency Ω given by Equation (101) as shown in Figure 8.9.

The angular velocity of the body given by

$$\vec{\Omega} \;=\; \hat{i}\Omega_1 + \hat{j}\Omega_2 + \hat{k}\Omega_3$$

can thus be written as

$$\vec{\Omega} \;=\; \vec{\Omega}_P + \Omega_3\hat{k} \tag{110}$$

with

$$|\vec{\Omega}| = \;\sqrt{\Omega_P^2 + \Omega_3^2}\; = \text{constant} \tag{111}$$

We find the angular velocity vector $\vec{\Omega}$ to have a constant magnitude and precessing about the axis of symmetry with the constant angular frequency Ω_0 (Refer to Figure 8.10). We find the vector $\vec{\Omega}$ to move on the surface of a cone about the axis of symmetry with constant angular frequency Ω_0. This motion takes place with respect to the principal axes of the body which are themselves rotating in space with angular frequency Ω. Equation (101) shows that closer the values of I_1 and I_3, lower becomes the precessional frequency Ω_0 compared to rotational frequency Ω_3. We may determine Ω_P and Ω_3 from a knowledge of the constant magnitudes of kinetic energy T and the angular momentum J given as

$$T = \;\frac{1}{2}I_1\Omega_p^2 + \frac{1}{2}I_3\Omega_3^2 \tag{112}$$

$$J^2 = \;I_1^2\Omega_P^2 + I_3^2\Omega_3^2 \tag{113}$$

The above results can be applied to the problem of rotation of the earth.

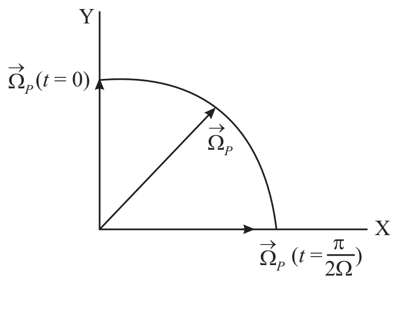

Fig. 8.9 *Symmetric Axis with Constant Frequencies*

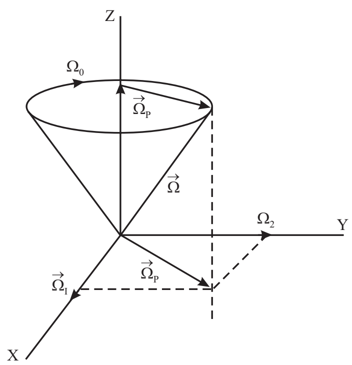

Fig. 8.10 *Symmetric Axis with Angular Frequencies*

NOTES

We know that the earth is almost symmetric about the north-south (polar) axis and slightly bulged at the equator. As a consequence we have I_1 slightly less than I_3. On calculation we obtain

$$\frac{I_3 - I_1}{I_1} = \frac{1}{306}$$

and $$\Omega_3 = \frac{2\pi}{24 \times 60 \times 60} \text{ rad s}^{-1}$$

The time period of precession of the axis of rotation of the earth is thus

$$T = \frac{2\pi}{\Omega_0} = \frac{2\pi}{\Omega_3}\frac{I_1}{I_3 - I_1} = \frac{2\pi}{\Omega_3} \times 306$$

or $$T = 306 \text{ days}$$

Thus, an observer on the earth should find the axis of rotation of the earth to trace out a circle about the north pole every 306 days which agrees well with observation.

8.4.6 Motion of Symmetric Top Under the Action of Gravity

Consider the motion of a **symmetric top** spinning about the axis of symmetry namely the Z′-axis of the body-fixed system. Z′-axis is taken to be one of the principal axes, the other two principal axes being X′ and Y′ axes.

According to the above consideration; the principal moments of inertia about the X′ and the Y′ axes, namely I_1 and I_2 are equal.

Let the top have its pivot at its lower tip O which is the common origin of the body fixed and the space-fixed coordinate system X′Y′Z′ and XYZ respectively.

Let G be the centre of gravity of the top at a distance l from the point O.

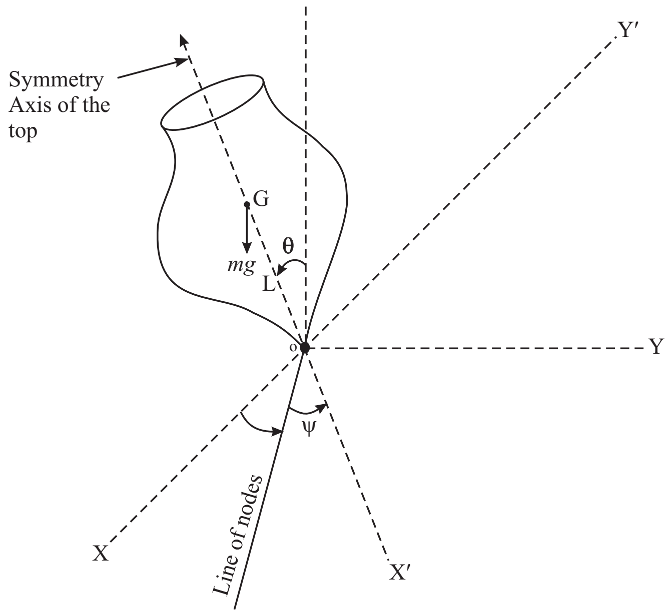

Fig. 8.11 *Symmetric Axis of Top under the Action of Gravity*

NOTES

The only force that acts on the top of mass *m* is *mg* which acts vertically downwards from the point G. Let us consider the Z-axis of the space-fixed system to pointing vertically upwards while the X and Y axes to lie in the horizontal plane (Refer to Figure 8.11).

The most convenient generalized coordinates in terms of which the motion of the top can be described are the Euler angles ϕ, θ and ψ as shown in the Figure 8.11.

The Lagrangian function for the top under consideration is given by

$$L = T - V = \frac{1}{2}I_1(\Omega_1^2 + \Omega_2^2) + \frac{1}{2}I_3\Omega_3^2 - mgl\cos\theta \tag{114}$$

Substituting for Ω_1, Ω_2 and Ω_3 in terms of Euler angles, the above becomes

$$L = \frac{1}{2}I_1\left[\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right] + \frac{1}{2}I_3\left[\dot{\psi} + \dot{\phi}\cos\theta\right]^2 - mgl\cos\theta \tag{115}$$

The above expression for *L* shows that ψ and ϕ are cyclic coordinates. As a consequence the momenta conjugate to these coordinates, namely p_y and p_f are constants of motion or the first integral of motion.

We have

$$p_y = \frac{\partial L}{\partial\dot{\psi}} = I_3[\dot{\psi} + \dot{\phi}\cos\theta] = I_3\Omega_3 = I_1 a \tag{116}$$

and $$p_f = \frac{\partial L}{\partial\dot{\phi}} = [I_1\sin^2\theta + I_3\cos^2\theta]\dot{\phi} + I_3\cos\theta\,\dot{\psi} = I_1 b \tag{117}$$

In the above the integrals of motion have been expressed in terms of new constants *a* and *b*.

A third integral of motion is the total energy of the top given by

$$E = T + V = \frac{1}{2}I_1[\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta] + \frac{1}{2}I_3\Omega_3^2 + mgl\cos\theta \tag{118}$$

Solving for $\dot{\phi}$ and $\dot{\psi}$, the equations (116) and (117) yield

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} \tag{119}$$

and $$\dot{\psi} = \frac{I_1 a}{I_3} - \left[\frac{b - a\cos\theta}{\sin^2\theta}\right]\cos\theta \tag{120}$$

Using Equation (119) and (120) in Equation (118) we get

$$E = \frac{1}{2}I_1^2 a^2 + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_1\frac{(b - a\cos\theta)^2}{\sin^2\theta} + mgl\cos\theta \tag{121}$$

For convenience we introduce a new quantity E' as

$$E' = E - \frac{1}{2}I_1^2 a^2 \tag{122}$$

Using Equation (121) in Equation (122) we obtain

$$E' = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_1\frac{(b-a\cos\theta)^2}{\sin^2\theta} + mgl\cos\theta \qquad (123)$$

We may consider E' to be the sum of the kinetic energy $\frac{1}{2}I_1\dot{\theta}^2$ and an effective potential energy function $V(\theta)$ defined as

$$V(\theta) = \frac{1}{2}I_1\frac{(b-a\cos\theta)^2}{\sin^2\theta} + mgl\cos\theta \qquad (124)$$

In view of Equation (124) we may write Equation (123) as

$$E' = \frac{1}{2}I_1\dot{\theta}^2 + V(\theta) \qquad (125)$$

The above gives

$$\dot{\theta} = \left[\frac{2}{I_1}\{E' - V(\theta)\}\right]^{1/2}$$

or

$$dt = \frac{d\theta}{\left[\frac{2}{I_1}\{E' - V(\theta)\}\right]^{1/2}}$$

Integrating the above between time limits $t = 0$ to $t = t$, we get

$$\int_0^t dt = \int_{\theta(0)}^{\theta(t)} \frac{d\theta}{\left[\frac{2}{I_1}\{E' - V(\theta)\}\right]^{1/2}} \qquad (126)$$

The solution of the above equation to get θ and hence solve for ϕ and ψ as functions of t is however complicated and involves elliptic integrals.

For this reason, only the qualitative features of the motion of top from energy considerations are presented as in motion under central force.

1. Steady Precession of the Top

The variation of the effective potential $V(\theta)$ with θ is as shown in Figure 8.12. We find $V(\theta)$ to assume infinite values for $\theta = 0$ and $\theta = \pi$ and and for a particular value, namely $\theta = \theta_0$, $V(\theta)$ assumes the minimum value. Thus the physically acceptable value of θ lies between 0 and π. The minimum value of $V(\theta)$ correspondes to the condition $\frac{\partial V(\theta)}{\partial\theta} = 0$. Using the expression for $V(\theta)$ given by Equation (124), the above condition yields

$$I_1 a\frac{b-a\cos\theta_0}{\sin\theta_0} - I_1\frac{(b-a\cos\theta_0)^2}{\sin^3\theta_0} - mgl\sin\theta_0 = 0$$

Solving the above we obtain

$$b - a\cos\theta_0 = \frac{a\sin^2\theta_0}{2\cos\theta_0}\left[1 \pm \sqrt{1 - \frac{4mgl\cos\theta_0}{I_1a^2}}\right] \qquad (127)$$

$b - a\cos\theta_0$ is a real quantity. Thus, for $\theta_0 < n/2$, we get

$$\sqrt{1 - \frac{4mgl\cos\theta_0}{I_1 a^2}} \geq 0$$

NOTES

or $$I_1^2 a^2 \geq 4\, mglI_1 \cos\theta_0$$

We have $$I_1 a = I_3\Omega_3.$$

$$I_3^2\Omega_3^2 \geq 4mgl\, I_1 \cos\theta_0$$

or $$\Omega_3 \geq \frac{2}{I_3}\sqrt{mgl\, I_1 \cos\theta_0} \qquad (128)$$

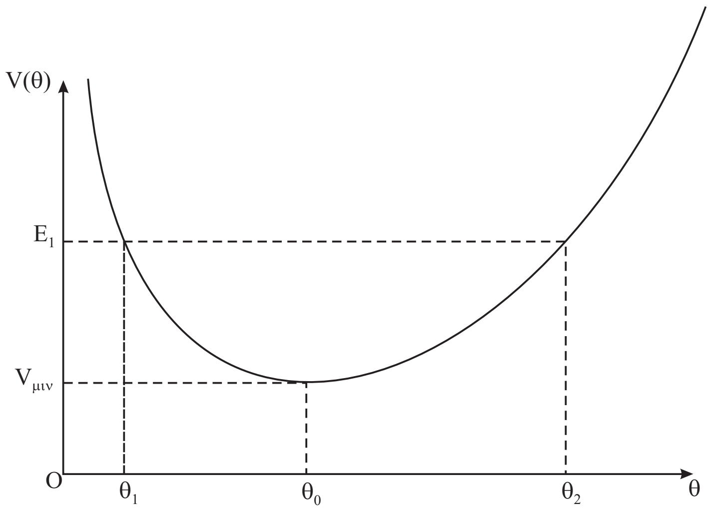

Fig. 8.12 *Steady Precession*

If the energy of the top is such that $E' = V_{\text{minimum}}$ then θ has only one value namely $\theta = \theta_0$. Thus corresponding to this energy the precession angle θ remain constants. This is referred to as **steady precession** in which case the symmetry axis of the top describes a right circular cone about the vertical Z-axis of the space-fixed system. The condition that must be satisfied for steady precession is that the value of Ω_3 be given according to Equation (128).

2. Nutational Motion of the Top

Referring to the effective potential energy diagram we find that if the energy of the top be such that $E' > V_{\text{minimum}}$, say $E = E_1'$, then the motion gets restricted between two values of θ, namely $\theta = \theta_1$ and $\theta = \theta_2$. This means that the symmetry axes OZ' of the rotating top varies its inclination θ with the vertical such that $\theta_1 \leq \theta \leq \theta_2$. This kind of motion of the top is referred to as **nutation**.

We have

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta}$$

Depending upon the values of θ_1 and θ_2, $\dot{\phi}$ may or may not change sign. If $\dot{\phi}$ does not change sign, the top precesses constantly about the vertical axis Z and the axis of rotation of the top, i.e., the Z' axis oscillates between $\theta = \theta_1$ and $\theta = \theta_2$. This is nutational motion which is an up and down motion of the top.

When the rotating top is released at the angle θ_1, it falls slightly due to the gravitational torque. As a consequence, it gains in the precessional as well as the nutational motion. The axis of the top reaches the maximum angle θ_2 with the vertical with the result of maximum precessional velocity and zero nutational velocity. The precessional velocity at $\theta = \theta_2$ can be obtained from the Equation (119) having the value

$$\dot{\phi} = \frac{I_3\Omega_3}{I_1} a = \frac{2mgl}{I_3\,\Omega_3}$$

The following are a few important results:

1. Frequency of nutational motion is given by

$$\Omega = \frac{I_3\Omega_3}{I_1} = a$$

2. Amplitude of nutation is given by

$$\theta_m = \frac{mgl\sin\theta_1}{I_3^2\,\Omega_3^2}$$

This shows that nutation is less if the top spins fast.

8.4.7 Illustrative Examples

Example 1: If the moment of inertia of a cube about an axis that passes through the centre of mass and the centre of any one face is I_0, find the moment of inertia of the cube about an axis passing through the centre of mass and one corner of the cube.

Solution:

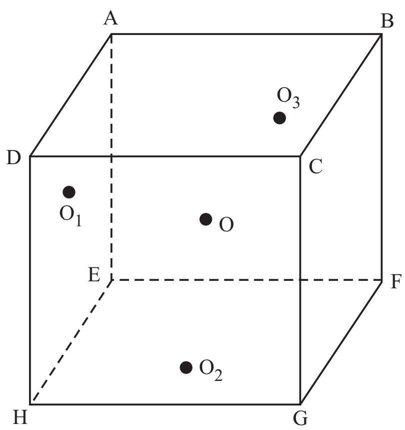

Fig. 8.13

NOTES

As shown in the figure, let O be the centre of mass of the cube ABCDEFGH. Considering the point O as the origin, let us choose a Cartesian coordinate system with axes X, Y and Z passing through the centres O_1, O_2 and O_3 of the three adjacent faces ADHF, DCGH and ABCD, respectively. We then have, according to the problem

$$I_{xx} = I_{yy} = I_{zz} = I_0 \tag{i}$$

and

$$I_{xy} = I_{yz} = I_{zx} = 0 \tag{ii}$$

Let an axis passing through the centre of mass of the cube and one corner, say C, have direction cosines α, β and γ, then the moment of inertia of the cube about this axis is given by

$$I = \alpha^2 I_{xx} + \beta^2 I_{yy} + \gamma^2 I_{zz} - 2\alpha\beta I_{xy} - 2\beta\gamma I_{yz} - 2\gamma\alpha I_{zx}$$

Using Equation (i) and (ii) in the above, we get

$$I = (\alpha^2 + \beta^2 + \gamma^2) I_0 \tag{iii}$$

The position vector of the corner C with respect to the origin O is given by

$$\vec{r} = \hat{i}a + \hat{j}a + \hat{k}a$$

Clearly, $$|\vec{r}| = \sqrt{3}a$$

We, thus, have

$$\alpha = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}, \quad \beta = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}, \quad \gamma = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}} \tag{iv}$$

Substituting Equation (*iv*) in Equation (*iii*), we obtain

$$I = \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) I_o = I_o$$

Example 2: A thin uniform circular disc of mass m and radius *r* lies in the *X–Y* plane (plane of the paper). A point mass $\frac{5m}{4}$ is attached to the rim of the disc as shown in the figure. The moment of inertia of the disc about an axis through its centre of mass and perpendicular to the plane of the paper is

$$I = \frac{mr^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Find the moment of inertia tensor of the system of the disc and point mass about the point A in the coordinate system shown.

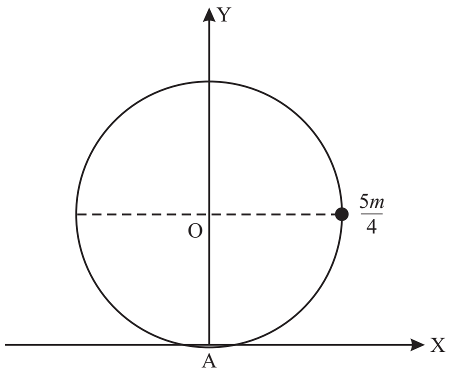

Fig. 8.14

Solution: The moment of inertia of the disc about the axes through the **centre of mass** O and perpendicular to the plane of the disc is given to be

$$I_{\text{cm}} = \frac{Mr^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad (i)$$

The mass point $\frac{5m}{4}$ attached to the disc contributes to the moments and products of inertia about the origin A is

$$I_{\text{ij}} = \frac{5m}{4}\left(r_o^2\delta_{ij} - x_i x_j\right) \qquad (ii)$$

where $\vec{r_0} = (x_1, x_2, x_3)$ is the radius vector of the mass point with respect to the origin A

In the above

$$\begin{aligned} i, j &= 1, 2, 3, \text{ and} \\ i, j &= 1, 2, 3 \text{ and} \\ \delta_{\text{ij}} &= 1 \text{ if } i = j \\ \delta_{\text{ij}} &= 0 \text{ if } i \neq j \qquad (iii) \\ x_1 &= x, x_2 = r \text{ and } x_3 = 0 \\ r_0^{\,2} &= x_1^{\,2} + x_2^{\,2} + x_3^{\,2} = 2r_0^{\,2} \end{aligned}$$

We, thus, get the moment of inertia tensor of the mass point about the point A as

$$\frac{I_{5m}}{4} = \frac{5m}{4}\begin{pmatrix} 2r^2 - r^2 & -r^2 & 0 \\ -r^2 & 2r^2 - r^2 & 0 \\ 0 & 0 & 2r^2 \end{pmatrix}$$

or

$$\frac{I_{5m}}{4} = \frac{5mr^2}{4}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad (iv)$$

NOTES

$$\delta_{ij} = 1 \text{ if } i = j \qquad (iii)$$

$$\delta_{ij} = 0 \text{ if } i \neq j$$

$$i, j = 1, 2, 3 \qquad (iv)$$

Using Equation (*ii*), (*iii*) and (*iv*), we obtain

$$I_{11} = m\left[\left(r_1^2 - x_{11}^2\right) + \left(r_2^2 - x_{21}^2\right) + \ r_3^2 - x_{31}^2\right) + \left(r_4^2 - x_{41}^2\right)\right]$$

or
$$I_{11} = m[(a^2 - a^2) + (a^2 - a^2) + (4a^2 - 0) + (4a^2 - 0)]$$

$$I_{11} = 8ma^2$$

Proceeding as above, we obtain

$$I_{12} = 0 = I_{13}$$

$$I_{21} = 0 = I_{23}$$

$$I_{31} = 0 = I_{32}$$

and
$$I_{22} = 2ma^2$$

$$I_{33} = 10ma^2$$

Thus, the matrix of the inertial tensor is

$$I = \begin{pmatrix} 8ma^2 & 0 & 0 \\ 0 & 2ma^2 & 0 \\ 0 & 0 & 10ma^2 \end{pmatrix}$$

Example 3. In Example 2 find the moment of inertia of the system about an axis which is inclined equally to the positive *X*-, *Y*- and *Z*-axes.

Solution: Since the axis is equally inclined to the positive *X*-, *Y*- and *Z*-axes, its direction cosines α, β and γ are equal.

The moment of inertia of the system about the axis is given in terms of the elements of the inertial tensor $I_{ij}(i, j = 1, 2, 3)$ as

$$I = \alpha^2 I_{11} + \beta^2 I_{22} + \gamma^2 I_{33} - 2\alpha\beta I_{12} - 2\beta\gamma I_{23} - 2\gamma_2 I_{31}$$

Using the values of I_{ij} obtained in Example 2 and taking

$$\alpha = \beta = \gamma$$

We get, from the above equation,

$$I = \alpha^2 8ma^2 + \alpha^2 2ma^2 + \alpha^2 10ma^2 = 20ma^2\alpha^2$$

We further have

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

$$3\alpha^2 = 1$$

or
$$\alpha^2 = \frac{1}{3}$$

Substituting for α^2, we obtain

$$I = 20ma^2 \times \frac{1}{3} = \frac{20}{3}ma^2$$

Example 4: Four point masses, each equal to m, are situated in the x–y plane at positions (a, 0, 0), (–a, 0, 0), (0, + 2a, 0) and (0, –2a, 0). The masses are joined by massless rods to form a rigid body.

Find the inertial tensor and express the tensor as a matrix using X-, Y- and Z-axes as the reference system.

Solution: The system to be considered is as shown in the figure. Let O be the origin of the reference system. The particles of mass m are situated at the points A(a, 0), B(–a, 0), C(0, 2a) and D(0, –2a).

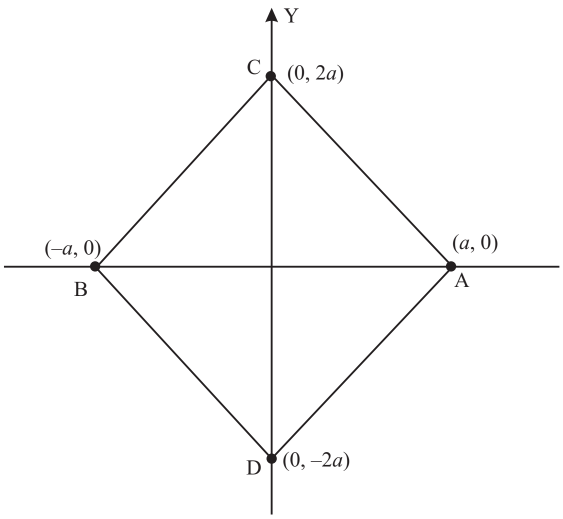

Fig. 8.15

By definition the elements I_{ij} of the inertial tensor are given by

$$I_{ij} = \sum_{k=1}^{4} m\left(r_k^2\delta_{ij} - xk_i xk_j\right)$$

or $$I_{ij} = m\left[\left(r_1^2\delta_{ij} - x_{1i}x_{1j}\right) + \left(r_2^2\delta_{ij} - x_{2i}x_{2j}\right) + \left(r_3^2\delta_{ij} - x_{3i}x_{3j}\right) + \left(r_4^2\delta_{ij} - x_{4i}x_{4j}\right)\right]$$

(i)

In the above,

$$\begin{aligned} r_1^2 &= x_{11}^2 + x_{12}^2 + x_{13}^2 = a^2 + 0 + 0 = a^2 \\ r_2^2 &= x_{21}^2 + x_{22}^2 + x_{23}^2 = (-a)^2 + 0 + 0 = a^2 \\ r_3^2 &= x_{31}^2 + x_{32}^2 + x_{33}^2 = 0 + (2a)^2 + 0 = 4a^2 \qquad (ii) \\ r_4^2 &= x_{41}^2 + x_{42}^2 + x_{43}^2 = 0 + (-2a)^2 + 0 = 4a^2 \end{aligned}$$

Example 5: Moments of inertia and products of inertia of a rigid body with respect to a coordinate system XYZ having the origin at some point within the body are I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz} and I_{zx}. Obtain the moment of inertia of the body about an axis inclined at angles α, β and γ with the X-, Y- and Z-axes, respectively.

NOTES

Solution: The figure shows the origin O within the rigid body and the coordinate system $X\ Y\ Z$. Let AB be the axis about which the moment of inertia of the body is to be found.

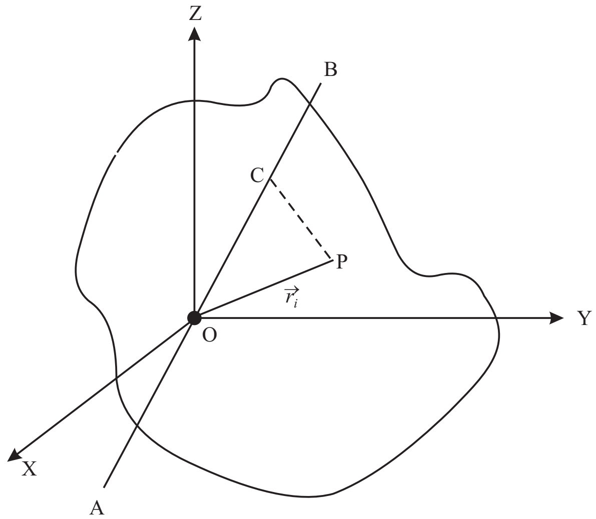

Fig. 8.16

Let the axis AB make angles of α, β and γ with the X-, Y- and Z-axes, respectively. If $\hat{n}$ be a unit vector along $\overrightarrow{OB}$, we get

$$\hat{n} = \hat{i}\cos\alpha + \hat{j}\cos\beta + \hat{k}\cos\gamma \qquad (i)$$

Let P be a particle of the body having mass m_i and radius vector $\vec{r}_i$.

Let PC be the perpendicular from P on the axis AB.

By definition, the moment of inertia of the particle about the axis AB $= m_i(PC)^2$.

Considering all the particles constituting the body, we get the moment of inertia of the body about the axis AB as

$$I = \sum m_i (PC)^2 = \sum m_i \left|\vec{r}_i \times \hat{n}\right|^2 \qquad (ii)$$

We have $$\vec{r}_i \times \hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & z_i \\ \cos\alpha & \cos\beta & \cos\gamma \end{vmatrix}$$

$$= \hat{i}\left(y_i\cos\gamma - z_i\cos\beta\right) + \hat{j}\left(z_i\cos\alpha - x_i\cos\gamma\right) + \hat{k}\left(x_i\cos\beta - y_i\cos\alpha\right)$$

The above gives

$$\left|\vec{r}_i \times \hat{n}\right|^2 = \left(y_i\cos\gamma - z_i\cos\beta\right)^2 + \left(z_i\cos\alpha - x_i\cos\gamma\right)^2 + \left(x_i\cos\beta - y_i\cos\alpha\right)^2$$

$$= \left(y_i^2 + z_i^2\right)\cos^2\alpha + \left(x_i^2 + z_i^2\right)\cos^2\beta + \left(y_i^2 + x_i^2\right)\cos^2\gamma \qquad (iii)$$
$$-2y_i z_i \cos\gamma\cos\beta - 2z_i x_i \cos\alpha\cos\gamma - 2x_i y_i \cos\beta\cos\alpha$$

Using Equation (*iii*) in (*ii*), we obtain

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$$I = \sum m_i\left(y_i^2 + z_i^2\right)\cos^2\alpha + \sum m_i\left(x_i^2 + z_i^2\right)\cos^2\beta + \sum m_i\left(y_i^2 + x_i^2\right)\cos^2\gamma$$
$$-2\sum m_i x_i y_i \cos\alpha\cos\beta - 2\sum m_i y_i z_i \cos\beta\cos\gamma - 2\sum m_i z_i x_i \cos\gamma\cos\alpha$$

Using the definitions of moments of inertia and products of inertia, the above can be expressed as

$$I = I_{xx}\cos^2\alpha + I_{yy}\cos^2\beta + I_{zz}\cos^2\gamma + 2I_{xy}\cos\alpha\cos\beta + 2I_{yz}\cos\beta\cos\gamma + 2I_{zx}\cos\gamma\cos\alpha$$

Using the theorem of parallel axes, the moment of inertia tensor disc about the axis through A (parallel to the axis through 0) is

$$I_A = I_{CM} + Mr^2$$

$$= \frac{mr^2}{4}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix} + Mr^2\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \qquad (iv)$$

$$= \frac{mr^2}{4}\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix} + \begin{pmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{pmatrix}\right]$$

or
$$I_A = \frac{mr^2}{4}\begin{pmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 6\end{pmatrix} \qquad (v)$$

In view of Equation (*iv*) and (*v*) the moment of inertia tensor of the disc and the mass point about the axis through *A* is

$$I = I_A = \frac{I_{5m}}{4}$$

$$= \frac{mr^2}{4}\begin{pmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 6\end{pmatrix} + \frac{5mr^2}{4}\begin{pmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$= \frac{mr^2}{4}\left[\begin{pmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 6\end{pmatrix} + \begin{pmatrix}5 & -5 & 0\\ -5 & 5 & 0\\ 0 & 0 & 10\end{pmatrix}\right]$$

or
$$I = \frac{mr^2}{4}\begin{pmatrix}10 & -5 & 0\\ -5 & 10 & 0\\ 0 & 0 & 16\end{pmatrix}$$

(*b*) The frequency of the circular motion is

$$\omega = \sqrt{\frac{k}{mr}} \qquad (vi)$$

$$U(q_1, q_2, ..., q_s) = U(q_{1o}, q_{2o}, ..., q_{s,o}) + \sum_k \left(\frac{\partial u}{\partial q_k}\right)_o q_k + \frac{1}{2!}\sum_{j,k}\left(\frac{\partial^2 u}{\partial q_j \partial q_k}\right)_o q_j q_k + ... \quad (vii)$$

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In the above expansion, we assume that $q_{0k} = 0$ and q_k is the displacement from $q_{0k} = 0$.

Using Equation (vi) in Equation (vii), we obtain

$$U(q_1, q_2, ..., q_s) = U(q_{1o}, q_{2o}, ..., q_{s,o}) + \frac{1}{2}\sum_{j,k}\left(\frac{\partial^2 u}{\partial q_j \partial q_k}\right)_o q_j q_k \quad (viii)$$

Without loss of generality, we can set

$$U(q_{10}, q_{20}, ..., q_{s,0}) = 0 \quad (ix)$$

Hence, we get from Equation (*viii*)

$$U(q_1, q_2, ..., q_s) = \frac{1}{2}\sum_{j,k}\left(\frac{\partial^2 u}{\partial q_j \partial q_k}\right)_o q_j q_k \quad (x)$$

or $$U(q_1, q_2, ..., q_s) = \frac{1}{2}\sum_{j,k} U_{jk} q_j q_k$$

where $$U_{jk} = \left(\frac{\partial^2 V}{\partial q_j \partial q_k}\right)_o$$

is constant depending upon the equilibrium values of $q_k's$, i.e., on $q_{ok}'s$.

Check Your Progress

5. What do you understand by the Eulerian angles?
6. What do you understand by herpolhode and polhode?

8.5 THE COMPOUND PENDULUM

Any rigid body irrespective of its shape or size capable of oscillating in a vertical plane about a horizontal axis passing through it is called a compound pendulum.

Consider a body *B* of mass '*m*' capable of oscillating in a vertical plane about the horizontal axis *XOX* passing through the point *O*. The point *O* is called the point of suspension. Let '*G*' be the position of the centre of gravity of the body. When the body is in its rest or equilibrium position the line joining *O* and *G* is vertical.

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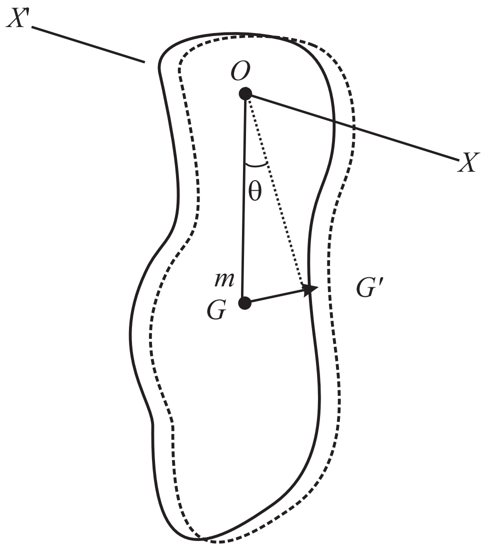

Fig. 8.17 *Motion in Compund Pendulum*

When the pendulum is displaced slightly from its equilibrium position to the dotted position, the centre of gravity undergoes an angular displacement from the point G to the point G' and the line OG turns to OG' by on angle θ.

At the displaced position, forces acting on the body are (i) mg acting vertically from G' and (ii) the reaction force mg acting vertically upwards from the point O. As a result a restoring torque acts tending to bring the pendulum to its equilibrium position.

If $OG = OG' = l$ then the restoring torque acting at the displaced position is given by

$$\tau_{\text{restoring}} = mg \times G'M$$

($G'M$ being the perpendicular distance between the two forces)

According to the figure we get

$$G'M = l \sin \theta$$

So that the restoring torque becomes

$$\tau_{\text{restoring}} = mg\, l \sin \theta$$

The torque applied in displacing the pendulum is

$$\tau_{\text{deflection}} = I\, \alpha$$

Where, I is the momenta of inertia of the pendulum about the axis $X'OX$ and α is the angular acceleration created in the pendulum.

For the equilibrium of the displaced position, we must have

$$\tau_{\text{deflection}} = -\tau_{\text{restoring}}$$

or $$I\, \alpha = -mg\, l \sin \theta$$

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or $$I\frac{d^2\theta}{dt^2} = -mg\,l\sin\theta$$

or $$\frac{d^2\theta}{dt^2} = \frac{-mgl}{I}\sin\theta$$

Considering θ to be small, we get sin θ = θ and hence

$$\frac{d^2\theta}{dt^2} = \frac{-mgl}{I}\theta$$

We find for small angular displacement, angular acceleration is directly proportional to angular displacement and is opposite to displacement.

As a consequence, the pendulum if lifts to itself at the displaced position undergoes angular simple harmonic motion.

The time period of the angular simple harmonic motion is given by

$$T = 2\pi\sqrt{\frac{\text{angular displacement}}{\text{angular acceleration}}}$$

Using the expression for angular acceleration obtained above, we get

$$T = 2\pi\sqrt{\frac{\theta}{d^2\theta/dt^2}} = 2\pi\sqrt{\frac{I}{mgl}}$$

If I_G be the momenta of inertia of the pendulum about an axis through G and parallel to $X'OX$ then we get according to the theorem of parallel axes

$$I = I_G + m\,l^2$$

If further 'k' be radius of gyration of the pendulum about the axis through G we get

$$I_G = m\,k^2$$

Thus

$$I = mk^2 + ml^2 = m(k^2 + l^2)$$

The time period of oscillation thus takes the form

$$T = 2\pi\sqrt{\frac{m\left(k^2+l^2\right)}{mgl}} = 2\pi\sqrt{\frac{k^2+l^2}{gl}}$$

or $$T = 2\pi\sqrt{\frac{l+\frac{k^2}{l}}{g}}$$

From a knowledge of l, k and T, we can calculate the acceleration due to gravity in the laboratory, which is the primary purpose of pendulum.

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We may find an alternative expression for time-period as follows:

Let us produce the OG upto a point S such that

$$GS = \frac{k^2}{l}$$

We then get

$$T = 2\pi\sqrt{\frac{OS + GS}{g}} = 2\pi\sqrt{\frac{L}{g}}$$

Where $L = OS + GS = l + \frac{k^2}{l}$

We thus find that the compound pendulum is equivalent to a simple pendulum of lenght L.

The points defined above is called the centre of oscillation. An important point to note that the centre of suspension and centre of oscillation are interchangeable. This means that if the pendulum is suspended through O or suspended through S, the time period of oscillation remains the same. This can be seen as below:

Let the pendulum be inverted and suspended through S. In this case, the time period of oscillation becomes

$$T' = 2\pi\sqrt{\frac{SG + \frac{k^2}{SG}}{g}}$$

Putting $SG = \frac{k^2}{l}$ we obtain

$$T' = 2\pi\sqrt{\frac{\frac{k^2}{l} + l}{g}} = T$$

Clearly, the length of the equivalent simple pendulum can be determined by finding experimentally the position of centre of oscillation corresponding to a given centre of suspension.

Examples of Compound Pendulum: In the laboratory various kinds of compound pendulum are in use. Some of them are (1) Bar pendulum (2) Kater's pendulum etc.

It is important to note that a compound pendulum is free from the limitations of a simple pendulum namely the plot of the pendulum as point mass, the suspension string as inelastic, flexible and mass less.

The working formula for a compound pendulum can also be obtained using Lagrangian or Hamiltonian formulation of mechanics.

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Check Your Progress

7. What is a compound pendulum?
8. Give some examples of compound pendulum.

8.6 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. If the distance between any pair of particles in a body remains fixed irrespective of its motion in space then the body is said to be a rigid body.
2. The angular velocity of rotation of the rigid body about an axis is independent of the choice of the origin of the body-fixed system provided the direction of the axis of rotation remains the same.
3. By definition, the position vector of the centre of mass of the body with respect to the space-fixed system is
$$\vec{R} = \frac{\sum m_i \vec{r_i}}{\sum m_i} = \frac{\sum m_i \vec{r_i}}{M}$$
where M is the total mass of the body.
4. The kinetic energy T of the rigid body described above is given by
$$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$
5. They refer to the angles corresponding to three successive rotations of the space-fixed system performed in a particular sequence or order, such that at the end, the axes of the space-fixed system coincide with those of the body-fixed system.
6. The curve traced out on the invariable plane by the point of contact with the ellipsoid is called herpolhode and the corresponding curve described on the ellipsoid is called polhode.
7. Any rigid body irrespective of its shape or size capable of oscillating in a vertical plane about a horizontal axis passing through it is called a compound pendulum.
8. In the laboratory various kinds of compound pendulum are in use. Some of them are (1) Bar pendulum (2) Kater's pendulum, etc.

8.7 SUMMARY

- A rigid body, irrespective of the number of particles of which it is made, has six degrees of freedom.

- $\vec{\Omega} = \frac{d\vec{\phi}}{dt}$ is the angular velocity of rotation of the body. The direction of $\vec{\Omega}$ is along the axis of rotation.
- We get the total angular momentum of the body about the origin of the space-fixed system as

$$\vec{J} = \sum_{i=1}^{N} m_i \left(\vec{r}_i \times \vec{v}_i \right)$$

- The principal axes of inertia are those with respect to which the off-diagonal elements of the inertia tensor vanish.
- If $I_1 \neq I_2 \neq I_3$, i.e., all the three principal moments of inertia be different, then the rigid body is said to be an asymmetric top.
- If two principal moments of Inertia be equal, i.e., $I_1 = I_2 \neq I_3$ the rigid body is said to be a symmetrical top.
- A rigid body is said to be a spherical top if all three principal moments of inertia be equal, i.e., $I_1 = I_2 = I_3$.
- The kinetic energy of the rotating rigid body relative to a coordinate system whose axes are the principal axes is given by

$$T_{\text{rot}} = \frac{1}{2}\vec{\Omega}\cdot\vec{J} = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)$$

- The time period of the angular simple harmonic motion is given by

$$T = 2\pi\sqrt{\frac{\text{angular displacement}}{\text{angular acceleration}}}$$

- The time period of oscillation thus takes the form

$$T = 2\pi\sqrt{\frac{m\left(k^2 + l^2\right)}{mgl}} = 2\pi\sqrt{\frac{k^2 + l^2}{gl}}$$

or

$$T = 2\pi\sqrt{\frac{l + \frac{k^2}{l}}{g}}$$

8.8 KEY WORDS

- **Rigid body:** If the distance between any pair of particles in a body remains fixed irrespective of its motion in space then the body is said to be a rigid body.
- **Angular momentum:** The product of the moment of inertia and the angular velocity is known as angular momentum.

- **Eulerian angles:** They refer to the angles corresponding to three successive rotations of the space-fixed system performed in a particular sequence or order, such that at the end, the axes of the space-fixed system coincide with those of the body-fixed system.
- **Compound pendulum:** Any rigid body irrespective of its shape or size capable of oscillating in a vertical plane about a horizontal axis passing through it is known as a compound pendulum.

8.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Describe angular velocity of a rigid body.
2. Write a short note on principal axes moments of inertia.
3. Describe Euler's equation for force free motion.
4. Briefly describe force free motion of a symmetrical rigid body.

Long-Answer Questions

1. Describe Angular momentum of a rigid body.
2. Give a detailed account of the Eulerian angles.
3. Discuss torque free motion of a rigid body.
4. Explain motion of symmetric top under the action of gravity.
5. Give a detailed account of the compound pendulum.

8.10 FURTHER READINGS

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BLOCK - IV
SMALL OSCILLATIONS AND NORMAL MODES

NOTES

UNIT 9 SPECIAL THEORY OF RELATIVITY

Structure

9.0 INTRODUCTION

Special theory of relativity is the generally accepted and experimentally well-confirmed physical theory regarding the relationship between space and time. It is the most accurate model of motion at any speed when gravitational effects are negligible. It implies a wide range of consequences, which have been experimentally verified, including length contraction, time dilation, relativistic mass, mass-energy equivalence, a universal speed limit and relativity of simultaneity. Time and space cannot be defined separately from each other. Rather, space and time are interwoven into a single continuum known as spacetime. Events that occur at the same time for one observer can occur at different times for another. In this unit you will explain theory of relativity discussing relativity in Newtonian mechanics and Galilean transformation. You will understand Einstein's idea and the basic postulates of special theory of relativity. Equivalence of space and time is also discussed this unit.

9.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain special theory of relativity
- Discuss Newtonian mechanics and Galilean transformation

- Understand Einstein's idea and the basic postulates of special theory of relativity
- Interpret equivalence of space and time

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9.2 SOME FUNDAMENTAL CONCEPTS: THEORY OF RELATIVITY

Relativity in Newtonian Mechanics

All motions involve displacement relative to something. According to Newton 'absolute motion' of a body is translation of the body from one 'absolute place' to another 'absolute place'. Newton, however, did not define the meaning of 'absolute place'. According to him, translatory motion of a body could be detected only in the form of motion relative to another material body.

Besides displacement, motion takes place with passage of time. Newton considered time as absolute or true meaning thereby that it proceeds uniformly being independent of place or position and is not affected by anything external.

If the position of a point is represented by the position vector $\vec{r}$ or the Cartesian coordinates (x, y z) and time is denoted by 't' then the set of four variables x, y, z and t specifies the position and the time at which same event occurs. This physical phenomenon can be represented by set of events involving different space-time coordinates. To fix up the position and time of an event we need a material, means together with the method of using them which is referred to as the space time frame of reference. It then becomes possible for an observer at rest in a given frame of reference to determine the position of an event at every instant of time.

Inertial Frame of Reference

According to Newton's first law of motion, a body tends to maintain its state of inertia (state of rest or of uniform motion) unless an unbalanced external force changes or tends to change its state of inertia. This statement becomes meaningless unless the motion is defined explicitly with respect to same frame of reference. To remove this situation, idea of 'absolute space' was introduced which essentially represents the reference frame relative to which all motions can be pefectly defined. This led to the choice of unaccelerated frames of reference because in such frame all the laws of mechanics retain the same form. By unaccelerated frame we mean a frame whose origin and coordinate axes are taken over a fixed star. For all practical purposes we may take our unaccelerated frame as a frame fixed over the earth's suface or a frame moving with a uniform velocity over the earth's surface. Such a coordinate frame in which Newton's first law of motion holds is called inertial frame of reference.

Frames which are accelerated relative to an inertial frame are non-inertial frames.

To conclude, an inertial frame may be defined to be the one with respect to which an isolated body (a body far removed from all other matter) would move with uniform velocity

Galilean Transformation

Galileo during his studies of projectile motion observed from different frames of reference obtained a set of equations which are called Galilean transformation equations. These equations can be obtained as follows:

Consider two co-ordinate frames S and S'

Let the coordinates of a point in space be x, y, z with respect to the frames and x', y' z' with respect to the frame S'. The two sets of coordinates are related as

$$x' = x'_0 + c_{11}x + c_{12}y + c_{13}z$$
$$y' = y'_0 + c_{21}x + c_{22}y + c_{23}z$$
$$z' = z'_0 + c_{31}x + c_{32}y + c_{33}z$$

The time t' and t in the two frames are related as

$$t' = t$$

In the above x'_0, y'_0, z'_0 are the coordinates of the origin of the frame S' with respect to the origin of the frame S.

c_{11} = the cosine of the angle between the x axis of frames S and S'

c_{12} = the cosine of the angle between the x axis of frames S and the y axis of the frame S'

c_{13} = the cosine of the angle between the x axis of frames S and the z axis of the frame S' and so on.

Let us now suppose that the axes x, y, z of the frame S parallel to the axes x, y, z of the frame S' respectively.

Let us further assume the frame S' to be moving with uniform velocity $\vec{v}$ with respect to the frame S and at $t = 0$, the origins of the two frames coincide.

We then get

$$x' = x - v_x\, t$$
$$y' = y - v_y\, t$$
$$z' = z - v_z\, t$$
$$t' = t$$

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Where v_x, v_y and v_z are components of $\vec{v}$ along the three axes of the motion of s' to be taking place along the x (or x') axis only the above equations reduce to

$$x' = x - v_x t$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

The above set of equations are known as the Galilean transformation equations. The derivation of the equations are based on the basic assumptions of classical mechanics namely

(i) The time 't' can be defined independently of any particular frame of reference and

(ii) The length is independent of frame of reference in which measurement is made.

In view of the above it becomes possible to choose one frame of reference and regard it to be absolutely at rest and consider the motion of all other systems with respect to it.

However, most careful observations have never revealed any anisotropy in physical space and as such the above conclusion does not appear to be acceptable.

Einstein's Idea and the Basic Postulates of Special Theory of Relativity

Newtonian Mechanics and Galilean transformation equations are based on the following concepts of time and distance.

(a) There exists an universal time 't' which can be defined independently of coordinate frame i.e., time is absolute.

(b) The distance between two simultaneous events taking place at two positions in space is an invariant quantity, the value of which is independent of the frame of reference with respect to which the measurement is made.

Besides the above, the laws of mechanics in all inertial systems and all inertial systems themselves are equivalent from the point of view of mechanics.

Einstein in 1905 proposed that motion through ether was a meaningless concept. According to him only relative motion i.e., motion relative to same frame of reference physical significance.

He rejected the two previous concepts mentioned above which recognises the principle that the physical laws should take the (1) same form

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in all frames reference. Special theory of relativity deals only with the systems in uniform translatory motion with respect to each other.

For accelerated systems, Einstein's general theory of relativity predicts correct results.

Einstein formulated the special theory of relativity on the basis of the following basic postulates:

Postulate 1: Any physical phenomenon should have the same form of development in all systems of inertia.

or

Any physical phenomenon have the same form of development in all inertial frames of reference.

The above postulate states the equivalence of all inertial frames. This is called the principle of relativity.

Postulate 2: The speed of light in free space is the same for all observers and is independent of the relative velocity between the source of light and the observer. This is called the principle of constancy of speed of light.

In the first reading it appears that the postulate 1 is similar to that for Newtonian relativity in which there exists an absolute frame of reference in which laws of motion are strictly valid. The experimental results of Michelson- Morley experiment however proved the non-existence of such absolute frame.

Hence Einstein did not introduce the concept of absolute frame of reference. He emphasized that the relativity principle as given in the above two postulates is valid in all frames of reference, which are in uniform rectilinear motion relative to one another and is applicable to the laws of mechanics as well as to the laws of electromagnetism.

It is important to note that postulate 2 demands a new set of transformation equations other than the Galilean transformation equations between two inertial frames of reference (x, y, z, t) and (x', y', z', t'). Such a set of equations was developed by H.A. Lorentz.

Check Your Progress

1. State Newton's first law of motion.
2. Define an inertial frame.
3. What do you understand by the Galilean transformation equations?
4. State the principle of constancy of speed of light.

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9.3 EQUIVALENCE OF SPACE AND TIME

In the theory of general relativity, the equivalence principle is the equivalence of gravitational and inertial mass. According to Albert Einstein's observation, the gravitational 'force' as experienced locally while standing on a gigantic body, such as the Earth, is the same as the pseudo-force experienced by an observer in a non-inertial (accelerated) frame of reference.

As per the Einstein's statement of the equality of inertial and gravitational mass, a little reflection will show that the law of the equality of the inertial and gravitational mass is equivalent to the assertion that the acceleration imparted to a body by a gravitational field is independent of the nature of the body. The Newton's equation of motion in a gravitational field can be expressed as:

(Inertial Mass) · (Acceleration) = (Intensity of the Gravitational Field) · (Gravitational Mass)

This is only possible when there is numerical equality between the inertial and gravitational mass that the acceleration is independent of the nature of the body. As per the equivalence principle given by Galileo in the early 17th century, the Galileo communicated through the experiment that the acceleration of a test mass due to gravitation is independent of the amount of mass being accelerated.

Kepler used the discoveries and postulates of Galileo to show that the equivalence principle can be accurately described through the real world example as what would occur if the moon were stopped in its orbit and dropped towards Earth. This can be deduced without knowing if or in what manner gravity decreases with distance, but requires assuming the equivalency between gravity and inertia.

During the Apollo 15 mission in 1971, astronaut David Scott showed that Galileo was right, as acceleration is the same for all bodies subject to gravity on the Moon, even for a hammer and a feather.

Newton's gravitational theory simplified and formalized Galileo's and Kepler's ideas by deducing from Kepler's planetary laws how gravity reduces with distance. For example, an inertial body moving along a geodesic through space can be trapped into an orbit around a large gravitational mass without ever experiencing acceleration. This is possible because space-time is radically curved in close vicinity to a large gravitational mass. In such a situation, the geodesic lines bend inward around the center of the mass and a free-floating (weightless) inertial body will simply follow those curved geodesics into an elliptical orbit. An accelerometer on-board would never record any acceleration.

By contrast, in Newtonian mechanics, gravity is assumed to be a force. This force draws objects having mass towards the center of any massive body. At the Earth's surface, the force of gravity is counteracted by the mechanical (physical) resistance of the Earth's surface. So in Newtonian physics, a person at rest on the surface of a (non-rotating) massive object is in an inertial frame of reference.

In physics, the space-time is any mathematical model that fuses the three dimensions of space and the one dimension of time into a single four-dimensional continuum. Space-time diagrams can be used to visualize relativistic effects, such as why different observers perceive where and when events occur.

Until the 20th century, the assumption had been that the three-dimensional geometry of the universe, i.e., its spatial expression in terms of coordinates, distances, and directions, was independent of one-dimensional time. However, in 1905, Albert Einstein specified the following special relativity on two postulates:

1. The laws of physics are invariant, i.e., identical in all inertial systems that is for non-accelerating frames of reference.
2. The speed of light in a vacuum is the same for all observers, regardless of the motion of the light source.

The logical consequence of these postulates together define the inseparable connection of the four dimensions, previously assumed as independent, of space and time.

Einstein framed his theory in terms of kinematics, the study of moving bodies. In 1908, Hermann Minkowski presented a geometric interpretation of special relativity that fused time and the three spatial dimensions of space into a single four-dimensional continuum now known as Minkowski space. A key feature of this interpretation is the formal definition of the space-time interval. Although measurements of distance and time between events differ for measurements made in different reference frames, the space-time interval is independent of the inertial frame of reference in which they are recorded.

Definition: ***Non-relativistic classical mechanics treats time as a universal quantity of measurement which is uniform throughout space and which is separate from space.***

Classical mechanics assumes that time has a constant rate of passage that is independent of the state of motion of an observer, or indeed of anything external. Furthermore, it assumes that space is Euclidean. According to the special relativity, time cannot be separated from the three dimensions of space, because the observed rate at which time passes for an object depends on the object's velocity relative to the observer. Additionally, the general

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relativity defines that how gravitational fields can slow the passage of time for an object as seen by an observer outside the field.

In ordinary space, a position is specified by three numbers which are termed as dimensions. In the Cartesian coordinate system, these are called x, y, and z. The position in space-time is termed as an event, and requires four numbers to be specified: the three-dimensional location in space and the position in time, as shown in Figure 9.1. Therefore, space-time is four dimensional. An event is something that happens instantaneously at a single point in space-time, represented by a set of coordinates x, y, z and t.

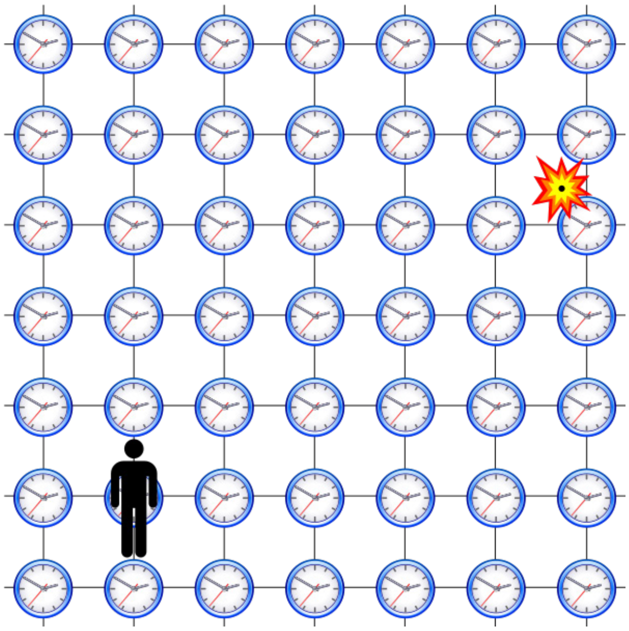

Fig. 9.1 *Position in Space-Time*

Figure 9.1 specifies that each location in space-time is marked by four numbers defined by a frame of reference, the position in space, and the time, which is shown as the reading of a clock located at each position in space. The 'observer' now synchronizes or harmonizes the clocks in accordance with their own reference frame.

Space-Time in Special Relativity

The space-time relativity can be defined through the following basic postulates.

Space-Time Interval: In three-dimensions, the distance Δd between two points can be defined using the Pythagorean Theorem as,

$$(\Delta d)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Even though two viewers may possibly measure the x, y, and z position of the two points using different coordinate systems, but the distance between

NOTES

the points will be the same for both assuming that they are measuring using the same units. The distance is considered 'invariant'.

In special relativity, however, the distance between two points is no longer the same if measured by two different observers when one of the observers is moving, because of Lorentz contraction. The situation is even more complicated if the two points are separated in **time** as well as in **space**. For example, if one observer sees two events occur at the same place, but at different times, a person moving with respect to the first observer will see the two events occurring at different places, because of their position as they are stationary, and the position of the event is receding or approaching. Accordingly, a different measure has to be used for measuring the effective 'distance' between two events.

The fundamental reason for merging space and time into space-time is that space and time are separately not invariant, which is to say that, under the proper conditions, different observers will disagree on the length of time between two events (because of time dilation) or the distance between the two events (because of length contraction). According to special relativity, a new invariant termed as space-time interval is defined which combines distances in space and in time.

Check Your Progress

5. Give Newton's equation of motion in a gravitational field.
6. What is meant by space-time?

9.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. According to Newton's first law of motion, a body tends to maintain its state of inertia (state of rest or of uniform motion) unless an unbalanced external force changes or tends to change its state of inertia.
2. An inertial frame may be defined to be the one with respect to which an isolated body (a body far removed from all other matter) would move with uniform velocity.
3. Galileo during his studies of projectile motion observed from different frames of reference obtained a set of equations which are called Galilean transformation equations. These equations are as follows:
4. The speed of light in free space is the same for all observers and is independent of the relative velocity between the source of light and the observer. This is called the principle of constancy of speed of light.

5. The Newton's equation of motion in a gravitational field can be expressed as:

   (Inertial Mass) · (Acceleration) = (Intensity of the Gravitational Field) · (Gravitational Mass)

6. In physics, the space-time is any mathematical model that fuses the three dimensions of space and the one dimension of time into a single four-dimensional continuum.

9.5 SUMMARY

- If the position of a point is represented by the position vector $\vec{r}$ or the Cartesian coordinates (x, y z) and time is denoted by 't' then the set of four variables x, y, z and t specifies the position and the time at which same event occurs.
- There exists an universal time 't' which can be defined independently of coordinate frame i.e., time is absolute.
- The distance between two simultaneous events taking place at two positions in space is an invariant quantity, the value of which is independent of the frame of reference with respect to which the measurement is made.
- Special theory of relativity deals only with the systems in uniform translatory motion with respect to each other.
- The gravitational 'force' as experienced locally while standing on a gigantic body, such as the Earth, is the same as the pseudo-force experienced by an observer in a non-inertial (accelerated) frame of reference.
- Non-relativistic classical mechanics treats time as a universal quantity of measurement which is uniform throughout space and which is separate from space.

9.6 KEY WORDS

- **Absolute motion:** According to Newton absolute motion of a body is translation of the body from one absolute place to another absolute place.
- **Special theory of relativity:** It is the generally accepted and experimentally well-confirmed physical theory regarding the relationship between space and time.

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- **Space-time:** In physics, the space-time is any mathematical model that fuses the three dimensions of space and the one dimension of time into a single four-dimensional continuum.
- **Space-time interval:** In three-dimensions, the distance Δd between two points can be defined using the Pythagorean Theorem as,

  $(\Delta d)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$

9.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Write short notes on the followings:
   a. Relativity in Newtonian mechanics
   b. Inertial frame of reference
   c. Equivalence of space and time
   d. Space-time in special relativity
2. Describe Galilean transformation briefly.
3. Discuss briefly Einstein's idea and the basic postulates of special theory of relativity.
4. Write a short note on space-time in special relativity.

Long-Answer Questions

1. Give a detailed account of the theory of relativity.
2. Explain equivalence of space and time.

9.8 FURTHER READINGS

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UNIT 10 LORENTZ TRANSFORMATION

Structure

10.0 INTRODUCTION

In physics, the Lorentz transformations are a one-parameter family of linear transformations from a coordinate frame in space time to another frame that moves at a constant velocity, the parameter, within the former. The transformations are named after the Dutch physicist Hendrik Lorentz. The respective inverse transformation is then parametrized by the negative of this velocity. In this unit you will understand Lorentz transformation equations and elementary property of Lorentz transformation. You will discuss consequences of Lorentz transformation. You will describe phenomenon of contraction of length and time dilation. You will interpret simultaneity of events, variation of mass with velocity and equivalence of mass and energy.

10.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain Lorentz transformation equations and its elementary property
- Discuss consequences of Lorentz transformation
- Describe simultaneity of events, variation of mass with velocity
- Understand equivalence of mass and energy

10.2 LORENTZ TRANSFORMATION EQUATIONS

NOTES

For the postulates of Einstein's special theory of relativity to be acceptable, it becomes important to inquire whether they are consistent with experimental observations or not. In order to make deductions from the special theory of relativity we must compare the descriptions of a given phenomenon in terms of two inertial frames moving relative to each other.

Let us consider two inertial frames I and I' having their coordinate axes parallel, i.e., the axes x, y, z of the frame I are respectively parallel to the axes x', y', z' of the frame I'.

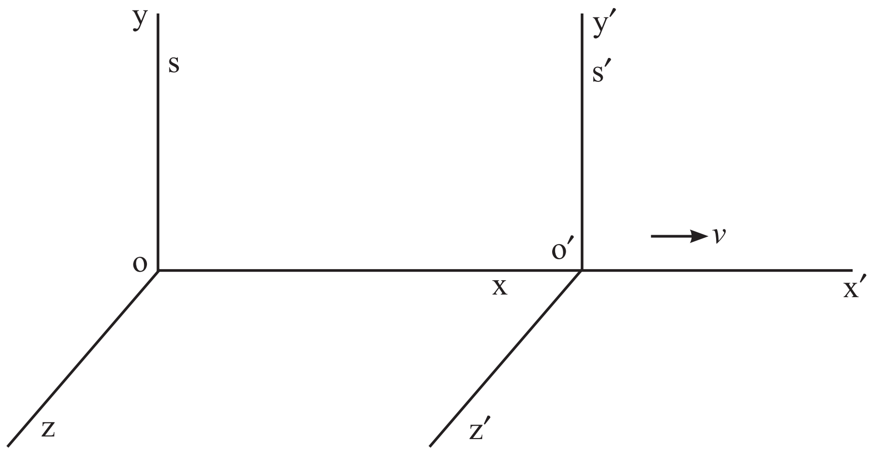

Fig. 10.1 *Relativeness of Two Frames*

Let the frame I' move with a uniform translational velocity $\vec{v}$.

Space-time coordinates of a point as observed from the frame I are (x, y, z, t) while those observed from the frame I' are (x', y', z', t').

If the motion of I' is restricted along the x' and hence the X axis then any measurement of the coordinates yield.

$$y' = y \tag{1}$$

$$z' = z \tag{2}$$

As velocity of light, say 'c' remains unaffected by the motion of medium a light signal starting from the origins o and o' at the time $t = 0 - t'$ when o and o' coincide spread out in spherical wave. After a time t, the spherical wave as observed from the frame I is described by the equation.

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{3}$$

The spherical wave as observed from the frame I' is described by the equation

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = 0 \tag{4}$$

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Equations (3) and (4) may for generality be written as

$$x^2 + y^2 + z^2 - c^2t^2 = s^2 \quad (5)$$

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = s'^2 \quad (6)$$

Since (x, y, z, t) and (x', y', z', t') have linear relations between them, we have, for any set of values of x, y, z, t which makes $s^2 = 0$, s'^2 will also be zero.

We may hence write

$$s'^2 = k\,(v)\,s^2 \quad (7)$$

where $k(v)$ is a constant being a function of v.

Alternatively if the frame I is in uniform translation motion with respect to the frame I' i.e., if the two frames interchange their roles then we have the relation.

$$s^2 = k\,(v)\,s'^2 \quad (8)$$

Equations (7) and (8) holds simultaneously when

$$k(v) = 1 \quad (9)$$

giving

$$s'^2 = s^2 \quad (10)$$

Using Equations (1), (2) and (10) we can write from Equations (5) and (6)

$$x'^2 - e^2t'^2 = x^2 - e^2t^2 \quad (11)$$

As x' and t' are linear functions of x and t only we may write

$$x' = \alpha\,(x^2 - vt) \quad (12)$$

and

$$t' = \beta t + \gamma x \quad (13)$$

Substituting Equations (12) in Equation (4) we get

$$\alpha^2(x^2 - 2vtx + v^2t^2) + y^2 + z^2 - c^2\,(\beta^2t^2 + 2\beta\gamma\, xt + \gamma^2x^2) = 0 \quad (14)$$

Equation (3) gives

$$y^2 + z^2 = c^2t^2 - x^2 \quad (15)$$

Using Equation (15) in (14) we obtain

$$\alpha^2x^2 - 2\alpha^2vtx + \alpha^2v^2t^2 + c^2t^2 - x^2 - c^2\,\beta^2t^2 - 2c^2\beta\gamma xt - c^2\gamma^2x^2 = 0$$

or

$$(\alpha^2 - c^2\gamma^2 - 1)\,x^2 - (2\alpha^2v + 2c^2\beta\gamma)\,xt + (\alpha^2v^2 - c^2\,\beta^2 + c^2)\,t^2 = 0 \quad (16)$$

Equation (16) is satisfied provided the coefficient of each term vanishes whence we get

$$\alpha^2 - c^2\gamma^2 - 1 = 0 \quad \text{or} \quad \alpha^2 - c^2\gamma^2 = 1 \quad (17)$$

$$\alpha^2v + c^2\beta\gamma = 0 \quad (18)$$

$$\alpha^2v^2 + c^2 = c^2\beta^2 \quad (19)$$

NOTES

Multiplying Equation (18) by v and subtracting (19) from it we obtain

$$vc^2\beta\gamma + c^2\beta^2 = c^2$$

or $$\beta\gamma = \frac{1-\beta^2}{v}$$

or $$\gamma = \frac{1-\beta^2}{\beta v} \qquad (20)$$

Substituting Equation (2)) in Equation (17) we get on rearranging the terms

$$v^2\alpha^2\beta^2 - c^2 + 2c^2\beta^2 - c^2\beta^4 = v^2\beta^2 \qquad (21)$$

Multiplying Equation (19) by β^2 we obtain

$$v^2\alpha^2\beta^2 + c^2\beta^2 = c^2\beta^4$$

or $$v^2\alpha^2\beta^2 - c^2\beta^4 = -c^2\beta^2 \qquad (22)$$

Using Equation (22) in Equation (21) we get

$$-c^2 - c^2\beta^2 + 2c^2\beta^2 = v^2\beta^2$$

or $$-(c^2 - c^2\beta^2) = -v^2\beta^2$$

or $$-c^2(1-\beta^2) = v^2\beta^2$$

or $$-\frac{(1-\beta^2)}{\beta^2} = \frac{v^2}{c^2}$$

or $$\beta^2 = \frac{1}{1-\dfrac{v^2}{c^2}}$$

or $$\beta = \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} \qquad (23)$$

Using Equation (23) in Equation (19) we obtain

$$\alpha^2 = \beta^2$$

Thus we have $$\alpha = \beta = \frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} \qquad (24)$$

and $$\gamma = \frac{-\beta v}{c^2} = -\frac{v/c^2}{\sqrt{1-\dfrac{v^2}{c^2}}} \qquad (25)$$

Using the above values α, β and γ we finally get

$$x' = x - \frac{vt}{\sqrt{1-\dfrac{v^2}{c^2}}} \qquad 26(a)$$

NOTES

$$y' = y \qquad 26(b)$$

$$z' = z \qquad 26(c)$$

$$t' = t - \frac{vx/c^2}{\sqrt{1-\frac{v^2}{c^2}}} \qquad 26(d)$$

We get by analogy the equations in the reverse case

$$x = x' + \frac{vt'}{\sqrt{1-\frac{v^2}{c^2}}} \qquad 27(a)$$

$$y = y' \qquad 27(b)$$

$$z = z' \qquad 27(c)$$

$$t = t' + \frac{vx'/c^2}{\sqrt{1-\frac{v^2}{c^2}}} \qquad 27(d)$$

The set of equations (26) and (27) are the direct and reverse Lorentz transformation equations of space and time coordinates in the two inertial frames under consideration.

An important special case arises when v is negligibly small compared to e i.e., when $\frac{v}{c} \to 0$.

In such a case we may neglect $\frac{v^2}{c^2}$ in comparison to 1 and the Equations (26) and (27) respectively take the forms

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \qquad (28)$$

and

$$\left.\begin{aligned} x &= x' + vt' \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}\right\} \qquad (29)$$

The above set of equations are the Galilean transformation equations.

Elementary Property of Lorentz Transformation

(A) Let us consider a second Lorentz transformation of the space time coordinates x', y', z', t' in the frame I' to x'', y'', z'', t'' in the frame I'' which is moving with a uniform velocity v' with respect to the frame I' along the $x'' = x' = x$ axis.

Proceeding in a manner similar to the previous section we obtain

$$x'' = \frac{1}{\sqrt{1-\frac{v''^2}{c^2}}}\left[x - v''t\right] \quad (30a)$$

$$y'' = y \quad (30b)$$

$$z'' = y \quad (30c)$$

$$t'' = \frac{1}{\sqrt{1-\frac{v''^2}{c^2}}}\left[t = \frac{v''x}{c^2}\right] \quad (30d)$$

where v'' is given by

$$v'' = \frac{v+v'}{1+\frac{vv'}{c^2}} \quad (31)$$

From the above results we may conclude that two successive Lorentz transformation is itself a Lorentz transformation.

(B) We have

$$v'' = \frac{v+v'}{1+\frac{vv'}{c^2}}$$

If v' be the velocity of light i.e., $v' = c$ we get from the above

$$v'' = \frac{v+c}{1+\frac{v}{c}} = \frac{c(v+c)}{v+c} = 0 \quad (32)$$

Thus the result of adding a velocity to the velocity of light is the velocity of light itself. Clearly the velocity of light in the maximum velocity that can be attained by any material body.

Hence the magnitude of velocity of light is the same in all coordinate frames. For this reason the velocity of light 'c' is called invariant i.e., it cannot be altered by Lorentz transformation.

Check Your Progress

1. Give the equation to describe the spherical wave as observed from the inertial frame *I*'.
2. Write the Galilean transformation equations.

10.3 CONSEQUENCES OF LORENTZ TRANSFORMATION

NOTES

There are two interesting cases in which the Lorentz transformation equations give results of special importance. These are contraction in space and time more commonly known as Length Contraction and Time Dilation.

(*i*) Phenomenon of Length Contraction

Consider two inertial frames of reference I and I'. Let us further assume the axes x, y, z of the frame I to be respectively parallel to the axes x', y', z' of the frame I' and the frame I' to be moving with a uniform $\vec{v}$ with respect to the frame I along the X' or the $-X$ axis.

Consider a rod of length 'l' connected rigidly in the frame I' parallel to the X' axis as shown in the figure:

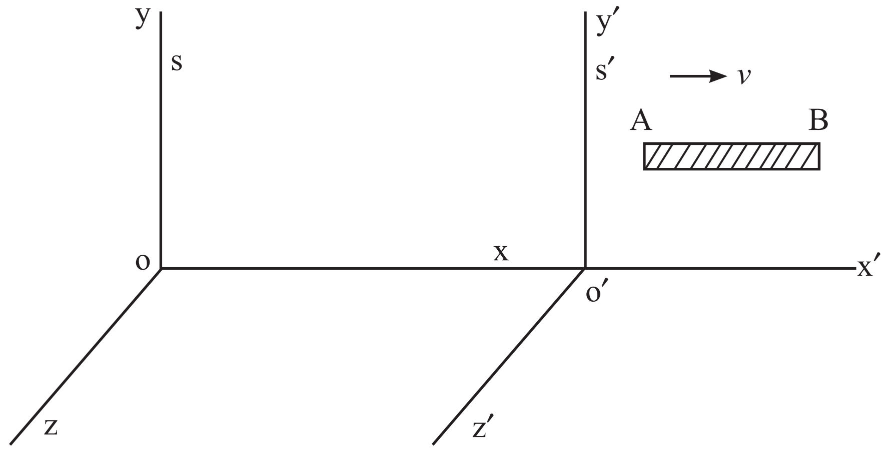

Fig. 10.2 *Length Contraction*

Let the end points A and B of the rod are defined by the coordinates x_1', 0, 0 and x_2', 0, 0 respectively.

The length l' of the rod as measured in the frame I' is

$$l' = x_2' - x_1'$$

If x_1, 0, 0 and x_2, 0, 0 be the coordinates of the end point A and B with respect to the frame I then the length of the rod as observed by an observer in the frame I will be

$$l = x_2 - x_1$$

According to Lorentz transformation equations we have

$$x_1 = \frac{x_1' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$x_2 = \frac{x_2' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

We thus have

$$l = x_2 - x_1 = \frac{x_2' + vt'}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{x_1' + vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$l = \frac{x_2' - x_1'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{l'}{\sqrt{1-\frac{v^2}{c^2}}}$$

The above relation gives

$$l' = l\left(1-\frac{v^2}{c^2}\right)^{1/2}$$

Thus to an observer at rest in an inertial frame *I*, the length of a rigid rod placed parallel to the *x*-axis of the frame I appears to be contracted by a factor $\sqrt{1-\frac{v^2}{c^2}}$ if the rod moves with a uniform velocity v along its length. The dimensions of the rod perpendicular to its direction of motion, however, remain unaffected because according to Lorentz transformation we have

$$y'_2 - y'_1 = y_2 - y_1$$
$$z'_2 - z'_1 = z_2 - z_1$$

The reverse situation is also true i.e., to an observer at rest in the frame *I′*, the rod fixed in the frame *I* appears to be contracted by the factor $\left(1-\frac{v^2}{c^2}\right)^{1/2}$ when the frame *I* moves with a velocity *v* with respect to the frame *I′*.

Thus we have the important result that "every rod appears longest if it is at rest with respect to a stationary observer. When the rod is moving it appears to be contracted by the factor $\left(1-\frac{v^2}{c^2}\right)^{1/2}$ in the direction of its motion while its dimensions perpendicular to the direction of motion remain unaltered.

(*ii*) Phenomenon of Time Dilation

Let us consider a clock rigidly connected with the inertial frame of reference I′ at same point (x', y', z').

Let t' be the time recorded by the clock.

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Then the time 't' as measured by an observer at rest in the inertial frame I when the frame I' moves with velocity relative to the frame I is given according to Lorentz transformation equation by

$$t = \left(t' + \frac{x'v}{c^2}\right) \Bigg/ \sqrt{1-\frac{v^2}{c^2}}$$

In the system I, the time interval $\Delta t = t_2 - t_1$ is thus related to the time interval $\Delta t' = t'_2 - t'_1$ as recorded by the clock as

$$\Delta t = t_2 - t_1$$

$$= \frac{\left(t_2' + \frac{x'v}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\left(t_1' + \frac{x'v}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$$

or
$$\Delta t = \frac{\left(t_2' + t_1'\right)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}}$$

Thus we get

$$\Delta t' = \Delta t\sqrt{1-\frac{v^2}{c^2}}$$

From the above results we find that the time interval $(t'_2 - t'_1)$ as recorded in the inertial frame I' is lengthened i.e., rate of the clock in I' gets slowed down as observed by an observer at rest in the frame I by a factor $\sqrt{1-\frac{v^2}{c^2}}$.

In the reverse case, same result is found to hold. From the above results we can conclude that a clock appears to go at its usual (fastest) rate when it is at rest relative to an observer fixed in an inertial frame. When the clock moves relative to the observer at rest in an inertial frame, its rate appears to be slowed down by the factor $\sqrt{1-\frac{v^2}{c^2}}$.

The above phenomenon is called time dilation.

There is an additional factor observed from I. Different I' clocks go at the same rate but with a phase constant depending on their positions. Further does a I' clock stand from the origin along the x-axis, the slower it appears to go.

Time dilation phenomenon has been verified experimentally by observing the decay phenomenon of nuclear particles such as π^+ meson.

Simultaneity of Events

NOTES

If two events are found to occur to an observer at rest at the same time then the events are said to be simultaneous. However, to an observer moving with a uniform velocity, the events do not appear to be simultaneous. We can hence say that simultaneity of events is not absolute but relative.

In order to explain simultaneity, let us consider two inertial frames I and I' having their x, y, z and x', y', z' respectively parallel. Let I' be moving with a uniform velocity v with respect to I along the x' or the x-axis.

Let in the frame I, two events occur simultaneously at two different space time points $(x_1, y_1, z_1\ t_1)$ and $(x_2, y_2, z_2\ t_2)$

Let t'_1 and t'_2 be the corresponding times for the events as recorded by an observer in the frame I'. We then have according to Lorentz transformation equations

$$t_1' = \frac{\left(t_1 - \dfrac{vx_1}{c^2}\right)}{\sqrt{1-\dfrac{v^2}{c^2}}}; \qquad t_2' = \frac{t_1 - \dfrac{vx_2}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

The above gives

$$t_2' - t_1' = \frac{\dfrac{v}{c^2}(x_1 - x_2)}{\sqrt{1-\dfrac{v^2}{c^2}}}$$

Since $\qquad x'_1 \neq x_2 \quad$ we get

$$t'_1 \neq t'_2$$

Thus the two events are no longer simultaneous to an observer in I' frame i.e., the frame moving relative to I frame.

Variation of Mass with Velocity

In non-relativistic mechanics (mechanics dealing with bodies having velocities very small compared to that of light), the inertial mass 'm' of a body defined according to $\vec{p} = m\vec{v}$ ($\vec{p}$ being the linear momentum of the body) is a constant equal to the mass when the body is at rest. In relativistic mechanics, which deals with bodies having velocities comparable with that of light it is expected that mass of a body be a function of its velocity, i.e., $m = m(v)$. The fuction $m(v)$ can be obtained easily considering collision between two bodies (particles).

Derivation of the Function *m* (*v*)

Consider two bodies each having mass m' moving in opposite directions with velocities v' and $-v'$ in an inertial frame I' along the x' axis as shown in the

Figure 10.3. Let us assume the collision to be perfectly inelastic so that after the collision the bodies stick together.

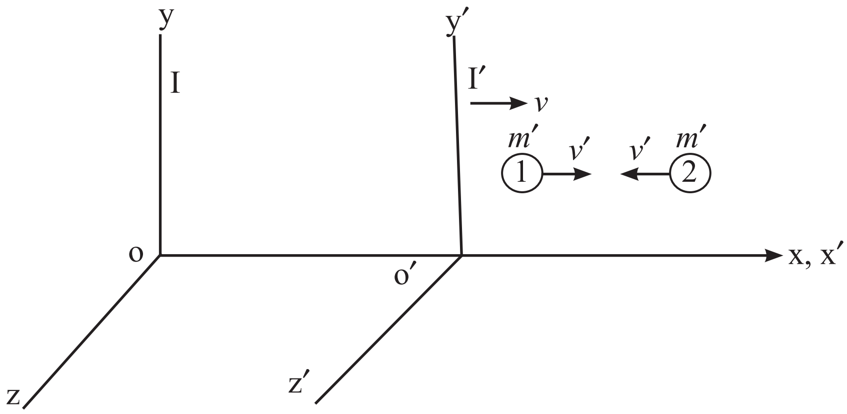

Fig. 10.3 *Deviatio of the Function m(v)*

Consider another inertial frame I with respect to which the frame I' is moving with a velocity v along the x' or the x-axis.

According to the principle of linear momentum we must have

$$m' u' \hat{i} + m\left(-u' \hat{i}\right) = \left(m_1 + m_2\right) \vec{v}$$

Clearly after the collision the system of bodies of mass $(m_1 + m_2)$ will be at rest with respect to the frame I'.

To an observer at rest in the frame I, the velocities of the colliding bodies are given by

$$v_1 = \frac{v' + v}{1 + \frac{v'v}{c^2}}, \quad v_2 = \frac{-v' + v}{1 - \frac{v'v}{c^2}} \tag{33}$$

Both v_1 and v_2 are along the x- axis. The above equations have been obtained using the relativistic law for addition of velocities.

Let with respect to the frame I, m and m_2 be the masses of the two bodies. Clearly after the collision the body formed has a mass $(m_1 + m_2)$ moving with the velocity v along the x-axis with respect to the frame I while it is at rest with respect to I'.

We get according to the law of conservation of linear momentum

$$m_1 v_1 + m_2 v_1 = (m_1 + m_2)\, v$$

Using Equations (33) in the above we get

$$m_1 \left[\frac{v' + v}{1 + \frac{v'v}{c^2}}\right] + m_2 \left[\frac{-v' + v}{1 - \frac{v'v}{c^2}}\right] = (m_1 + m_2)\, v \tag{34}$$

Dividing Equation (34) by m_2 and simplifying we obtain

$$\frac{m_1}{m_2} = \frac{1+\dfrac{v'v}{c^2}}{1-\dfrac{v'v}{c^2}} \tag{35}$$

We have

$$1-\frac{v_1^2}{c^2} = 1-\frac{1}{c^2}\left[\frac{v'+v}{1+\dfrac{v'v}{c^2}}\right]^2$$

Simplifying the above we obtain

$$1-\frac{v_1^2}{c^2} = \frac{\left(1-\dfrac{v^2}{c^2}\right)\left(1-\dfrac{v'^2}{c^2}\right)}{\left(1+\dfrac{v'v}{c^2}\right)^2}$$

or

$$\sqrt{1-\frac{v_1^2}{c^2}} = \frac{\sqrt{\left(1-\dfrac{v^2}{c^2}\right)\left(1-\dfrac{v'^2}{c^2}\right)}}{1+\dfrac{v'v}{c^2}}$$

or

$$1+\frac{v'v}{c^2} = \frac{\sqrt{\left(1-\dfrac{v^2}{c^2}\right)\left(1-\dfrac{v'^2}{c^2}\right)}}{\sqrt{1-\dfrac{v_1^2}{c^2}}} \tag{36}$$

Proceeding in a similar manner we obtain

$$1-\frac{v'v}{c^2} = \frac{\sqrt{\left(1-\dfrac{v^2}{c^2}\right)\left(1-\dfrac{v'^2}{c^2}\right)}}{\sqrt{1-\dfrac{v_2^2}{c^2}}} \tag{37}$$

Substituting Equations (36) and (37) in Equation (35) we get

$$\frac{m_1}{m_2} = \frac{\sqrt{1-\dfrac{v_2^2}{c^2}}}{\sqrt{1-\dfrac{v_1^2}{c^2}}} \tag{38}$$

If $v_2 = 0$ i.e., if the body 2 is at rest with respect to the frame I then its mass can be set equal to the rest mass m_0.

Let as further consider $v_1 = v$ i.e., velocity of the body 1 with respect to the frame be v. We can write $m_1 = m$. We then get from Equation (38)

$$\frac{m}{m_0} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

NOTES

or $$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} \tag{39}$$

Equation (39) gives the mass of a body when it moves with a velocity v and is the required relation between the mass of a body and its velocity.

We find from Equation (39) that variation of mass of a body moving with velocities small compared to that of light is insignificant. The variation becomes significant when the velocity of the body becomes comparable to that of light. This is illustrated in the Figure 10.4.

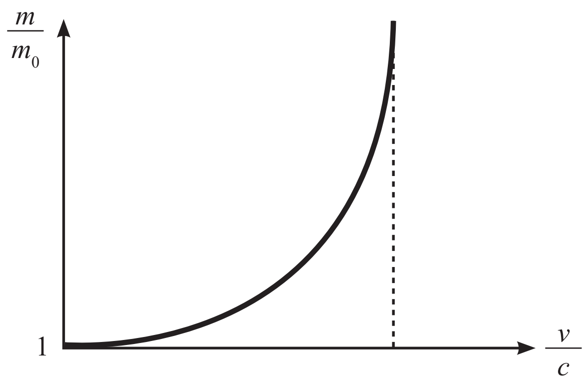

Fig. 10.4 *Variation in the Velocity*

Equivalence of Mass and Energy

Consider a body moving under the action of a force. If F be the component of the force along the direction of infinitesimal displacement dr then if the body starts from rest, the kinetic energy gained by the body is given by

$$T = \int_0^r F\,dr\,;$$ r is the distance over which the force acts

If m be the man and v be its velocity at the instant of time t we get

$$F = \frac{d}{dt}(mv)$$

Thus $$T = \int_0^r \frac{d}{dt}(mv)\quad dr = \int_0^{mv} v\,d(mv)$$

NOTES

or $$T = \int_0^v v\, d\left[\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\right]$$

Integrating by parts the above gives

$$T = \frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 \int_0^v \frac{v\, dv}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} + m_0 c^2 \left\{\sqrt{1-\frac{v^2}{c^2}}\right\}_0^v$$

or $$T = \frac{m_0 v^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0 c^2$$

or $$T = m_0 c^2 - m_0\, c^2 \qquad (40) \left[\text{using } m = \frac{m_o}{\sqrt{1-\frac{v^2}{c^2}}}\right]$$

The above may be written as

$$T = (\Delta\, m)\, c^2$$

Where $\Delta\, m = m - m_0$

Equation (40) gives

$$mc^2 = T + m_0\, c^2$$

If E be the total energy of the body i.e., $E = mc^2$ we find the energy of the body when at rest to be equal to $E_0 = m_0 c^2$. E_0 is called the rest energy. We hence obtain

$$E = E_0 + T$$

$E_0 = m_0 c^2$ implies that mass is a form of energy, thus a mass m is equivalent to an energy $E = mc^2$.

Check Your Progress

3. What do you understand by the phenomenon of time dilation?
4. What is rest energy?

10.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

NOTES

1. The spherical wave as observed from the frame I' is described by the equation

   $x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$

2. The following set of equations are the Galilean transformation equations:

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned}\right\} \qquad (1)$$

and

$$\left.\begin{aligned} x &= x' + vt' \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}\right\} \qquad (2)$$

3. A clock appears to go at its usual (fastest) rate when it is at rest relative to an observer fixed in an inertial frame. When the clock moves relative to the observer at rest in an inertial frame, its rate appears to be slowed down by the factor $\sqrt{1-\frac{v^2}{c^2}}$.

   The above phenomenon is called time dilation.

4. If E be the total energy of the body i.e., $E = mc^2$ we find the energy of the body when at rest to be equal to $E_0 = m_0c^2$. E_0 is called the rest energy. We hence obtain

10.5 SUMMARY

- If the frame I is in uniform translation motion with respect to the frame I' i.e., if the two frames interchange their roles then we have the relation.
- Two successive Lorentz transformation is itself a Lorentz transformation.
- The velocity of light 'c' is called invariant i.e., it cannot be altered by Lorentz transformation.
- Every rod appears longest if it is at rest with respect to a stationary observer. When the rod is moving it appears to be contracted by the

factor $\left(1-\frac{v^2}{c^2}\right)^{1/2}$ in the direction of its motion while its dimensions perpendicular to the direction of motion remain unaltered.

- When the clock moves relative to the observer at rest in an inertial frame, its rate appears to be slowed down by the factor $\sqrt{1-\frac{v^2}{c^2}}$.

NOTES

10.6 KEY WORDS

- **Lorentz transformations:** Lorentz transformations are a one-parameter family of linear transformations from a coordinate frame in space time to another frame that moves at a constant velocity, the parameter, within the former.
- **Simultaneity of events:** If two events are found to occur to an observer at rest at the same time then the events are said to be simultaneous. Simultaneity of events is not absolute but relative.
- **Time dilation:** It is a difference in the elapsed time measured by two observers, either due to a velocity difference relative to each other, or by being differently situated relative to a gravitational field.
- **Rest energy:** The energy of the mass component of a body at rest; equal to the product of its rest mass and the square of the speed of light is called rest energy.

10.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Describe elementary property of Lorentz transformation.
2. Deduce the relation between the mass of a body and its velocity.
3. Write a short note on equivalence of mass and energy.
4. Describe the phenomenon of length contraction briefly.
5. Give a brief account of phenomenon of time dilation.

Long-Answer Questions

1. Discuss Lorentz transformation equations.
2. Explain consequences of Lorentz transformation.

10.8 FURTHER READINGS

NOTES

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UNIT 11 ONE DIMENSIONAL OSCILLATOR

NOTES

Structure

11.0 INTRODUCTION

A quantum harmonic oscillator is the quantum-mechanical analog of the classical harmonic oscillator. The energy possessed by an object because of its position relative to other objects, stresses within itself, its electric charge, or other factors is known as potential energy. This energy is related with forces that act on a body in a manner that the total work done by these forces on the body relies only upon the initial and final positions of the body in space. These forces known as conservative forces, can be represented at every point in space by vectors exhibited as gradients of a certain scalar function called potential. A rigid body is believed to be in equilibrium if there is no change in translational motion and no change in rotational motion. In this unit you will study potential energy and equilibrium. You will describe one dimensional oscillator. Stable, unstable and neutral equilibrium is also discussed.

11.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand potential energy and equilibrium
- Explain motion of a one dimensional oscillator
- Define stable, unstable and neutral equilibrium
- Calculate the total energy of the oscillator

11.2 POTENTIAL ENERGY AND EQUILIBRIUM

NOTES

Let us consider a conservative system described by s generalized coordinates q_1, ..., q_s. The potential energy U of the system is then a function of only the coordinates, i.e.,

$$U = U(q_1, ..., q_s) \tag{1}$$

The generalized forces acting on the system can be derived from the potential function U according to

$$Q_k = -\frac{\partial U}{\partial q_k}; \qquad k = 1, 2, ..., s \tag{2}$$

The system is said to be in equilibrium if all the generalized forces vanish. Thus, for equilibrium we have

$$Q_k = 0 \tag{3}$$

or

$$\left(\frac{\partial U}{\partial q_k}\right)_o = 0; \qquad k = 1, 2, ..., s \tag{4}$$

In the above, the suffix o indicates equilibrium. We find that at the equilibrium configuration of the system described by the coordinates q_{10}, q_{20}, ..., q_{s0}, the potential energy has an extremum value. If, at $t = 0$, the system exists in the equilibrium configuration and all the generalized velocities $\dot{q}_{ko}\,(k = 1, 2, ..., s)$ are zero, the system remains in the equilibrium configuration indefinitely.

Check Your Progress

1. What is potential energy of a conservative system?
2. When is the system said to be in equilibrium?

11.3 ONE DIMENSIONAL OSCILLATOR: STABLE, UNSTABLE AND MISSING EQUILIBRIUM

Let us consider the motion of a one-dimensional oscillator. Clearly, the oscillator has only one degree of freedom.

Let q be the generalized coordinate that describes the oscillator. The potential energy of the oscillator is a function of only the coordinate q, i.e.,

$$U = U(q) \tag{5}$$

Let the value of q be q_o in the equilibrium state of the oscillator.

Considering a small disturbance from the equilibrium configuration we can expand $U(q) - U(q_0)$ in powers of $(q - q_0)$ as a Taylor's series

$$U(q) - U(q_0) = (q-q_o)\left(\frac{dU(q)}{dq}\right)_o + \frac{1}{2!}(q-q_o)^2\left(\frac{d^2U(q)}{dq^2}\right)_o + ...$$

or
$$U(q) = U(q_o) + \left(\frac{dU(q)}{dq}\right)_o (q-q_o) + \frac{1}{2!}\left(\frac{d^2U(q)}{dq^2}\right)_o (q-q_o)^2 + ... \quad (6)$$

Under the equilibrium condition of the oscillator the generalized force Q is zero, i.e.,

$$Q = \left(\frac{dU(q)}{dq}\right)_o = 0 \quad (7)$$

Using Equation (7) in Equation (6), we obtain

$$U(q) = U(q_o) + \frac{1}{2}\left(\frac{d^2U(q)}{dq^2}\right)_o (q-q_o)^2 \quad (8)$$

The potential energy $U(q_0)$ in the equilibrium configuration can be conveniently set equal to zero by considering the minimum potential energy in the stable equilibrium condition to be equal to zero. We can then write Equation (8) as

$$U(q) = \frac{1}{2}\left(\frac{d^2U(q)}{dq^2}\right)_o (q-q_o)^2 \quad (9)$$

Considering the origin of coordinates at $q_o = 0$, we may write the potential energy as

$$U(q) = \frac{1}{2}kq^2 \quad (10)$$

where
$$k = \left(\frac{d^2U(q)}{dq^2}\right)_o \quad (11)$$

k being a positive parameter at the stable equilibrium configuration.

The kinetic energy of the oscillator is a homogeneous quadratic function of the generalized velocity

$$T = \frac{1}{2}\alpha(q)\dot{q}^2 \quad (12)$$

In the above, $\alpha(q)$ is a function of coordinate q and like $U(q)$ can be expanded in powers of $(q - q_0)$ as a Taylor's series about the equilibrium configuration

$$\alpha(q) = \alpha(q_o) + \left(\frac{d\alpha(q)}{dq}\right)_o (q-q_o) + \frac{1}{2!}\left(\frac{d^2\alpha(q)}{dq^2}\right)_o (q-q_o)^2 + ... \quad (13)$$

NOTES

Since Equation (12) is quadratic in $(q - q_0) = q$, the lowest approximation to T is obtained by retaining only the first term in the series given by Equation (13). We may thus write Equation (12) as

$$T = \frac{1}{2}\alpha(q_o)q^2 \tag{14}$$

Putting $\alpha(q_0) = m$, we obtain

$$T = \frac{1}{2}m\dot{q}^2 \tag{15}$$

Using Equation (10) and (15) we may write the Lagrangian of the oscillator, in the limit of small oscillation about the stable equilibrium configuration, as

$$L = T - U = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2 \tag{16}$$

The Lagrange's equation of motion for the oscillator is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \tag{17}$$

From Equation (16) we get

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} \; ; \frac{\partial L}{\partial q} = -kq$$

Substituting the above in Equation (17) we obtain

$$\frac{d}{dt}(m\dot{q}) = -kq$$

or $$m\ddot{q} + kq = 0$$

or $$\ddot{q} + \omega^2 q = 0 \tag{18}$$

where $$\omega = \sqrt{\frac{k}{m}} \tag{19}$$

The general solution of Equation (18) can be written in the form

$$q = A_1 \sin \omega t + A_2 \cos \omega t$$

The above can be alternatively expressed in the form

$$q = A \cos(\omega t + \alpha) \tag{20}$$

where $$A_1 = -A \sin \alpha, \; A_2 = A \cos \alpha$$

so that the amplitude of oscillation A is given by

$$A = \sqrt{A_1^2 + A_2^2} \tag{21}$$

and the initial phase or the epoch is given by

$$\tan \alpha = -\frac{A_1}{A_2} \tag{22}$$

The time period of oscillation is given by

$$T = \frac{2\pi}{\omega} \tag{23}$$

The velocity of the oscillator is

$$v = \dot{q} = -A\omega\sin(\omega t + \alpha) \quad (24)$$

so that the total energy of the oscillator is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kq^2$$

$$= \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \alpha) + \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \alpha)$$

or

$$E = \frac{1}{2}mA^2\omega^2 = \text{a constant} \quad (25)$$

We find that the total energy of the oscillator is a constant of motion being proportional to the square of the amplitude and to the square of the frequency of the oscillator. As the system oscillates, the energy changes from kinetic to potential and vice versa.

11.3.1 Stable Equilibrium

The equilibrium is said to be stable if the system when displaced slightly from its equilibrium configuration and left to itself returns to the equilibrium configuration. In other words, the equilibrium is stable if the system undergoes small bounded motion about the equilibrium configuration when displaced slightly from the equilibrium configuration and left to itself.

11.3.2 Unstable Equilibrium

The equilibrium is said to be unstable in a situation when the system is displaced slightly from its equilibrium configuration and is left to itself, and the displacement from the equilibrium configuration goes on increasing. In other words, equilibrium is unstable if the result of a small displacement given to the system in its equilibrium configuration is an unbounded motion.

It may be noted that under the stable equilibrium condition, the extremum value of U is the minimum, while it is the maximum under the unstable equilibrium condition.

11.3.3 Neutral Equilibrium

The neutral equilibrium is a kind of equilibrium of a body specifically placed such that when moved slightly it neither tends to return to its former position not depart more widely from it, as a perfect sphere or cylinder on a horizontal plane. As per the standard definition, "If a body remains in its new position when disturbed from its previous position, it is said to be in a state of neutral equilibrium." The example of neutral equilibrium state can be given by a ball and which is placed on a horizontal surface. When the ball is rolled over the surface then it is displaced from its previous position. It now remains in its new position and does not return to its previous position. This is termed as neutral equilibrium.

NOTES

We can state the neutral equilibrium with the help of the following equation,

$$\frac{du}{dx} = \frac{d^2u}{dx^2} = 0$$

When the particle is moved slightly away from any given point and it still remains in equilibrium, i.e., it will neither return to its initial state nor will it continue to move then the neutral equilibrium point corresponds to a flat spot in a $U(x)$ curve.

Figure 11.1 illustrates the different types of equilibrium points, i.e., stable, unstable and neutral.

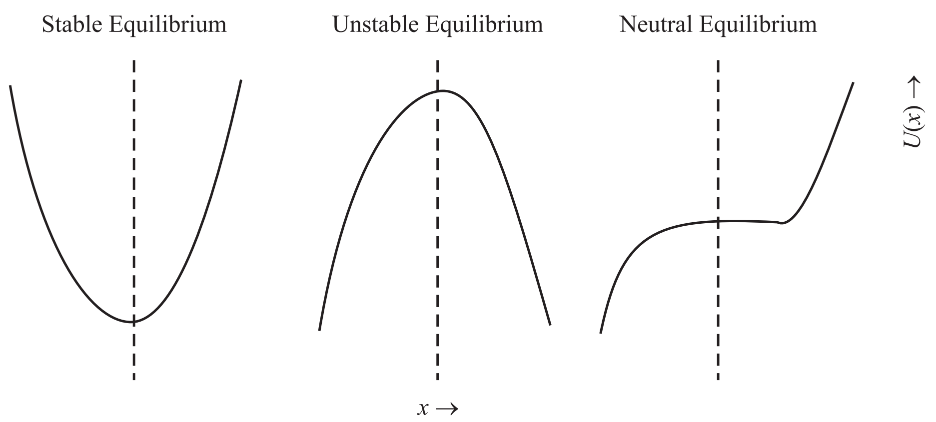

Fig. 11.1 *Different Types of Equilibrium Points*

Check Your Progress

3. Write the Lagrange's equation of motion for the oscillator.
4. What is the total energy of the oscillator?
5. Define stable equilibrium.
6. What do you mean by unstable equilibrium?

11.4 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Let us consider a conservative system described by s generalized coordinates $q_1, \ldots.., q_s$. The potential energy U of the system is then a function of only the coordinates, i.e.,

$$U = U(q_1, \ldots.., q_s)$$

2. The system is said to be in equilibrium if all the generalized forces vanish.
3. The Lagrange's equation of motion for the oscillator is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \tag{17}$$

4. The total energy of the oscillator is a constant of motion being proportional to the square of the amplitude and to the square of the frequency of the oscillator. As the system oscillates, the energy changes from kinetic to potential and vice versa.
5. The equilibrium is said to be stable if the system when displaced slightly from its equilibrium configuration and left to itself returns to the equilibrium configuration.
6. The equilibrium is said to be unstable in a situation when the system is displaced slightly from its equilibrium configuration and is left to itself, and the displacement from the equilibrium configuration goes on increasing.

NOTES

11.5 SUMMARY

- The generalized forces acting on the system can be derived from the potential function U according to

$$Q_{\text{k}} = -\frac{\partial U}{\partial q_k}\ ; \qquad k = 1, 2, \ldots.., s$$

- The equilibrium is stable if the system undergoes small bounded motion about the equilibrium configuration when displaced slightly from the equilibrium configuration and left to itself.
- Equilibrium is unstable if the result of a small displacement given to the system in its equilibrium configuration is an unbounded motion.
- The kinetic energy of the oscillator is a homogeneous quadratic function of the generalized velocity

$$T = \frac{1}{2}\alpha(q)\dot{q}^2$$

- If a body remains in its new position when disturbed from its previous position, it is said to be in a state of neutral equilibrium.

11.6 KEY WORDS

NOTES

- **Potential energy:** The energy in an object due to its position is known as potential energy.
- **Equilibrium:** The system is said to be in equilibrium if all the generalized forces vanish.
- **Stable equilibrium:** The equilibrium is stable if the system undergoes small bounded motion about the equilibrium configuration when displayed slightly from the equilibrium configuration and left to itself.
- **Unstable equilibrium:** Equilibrium is unstable if the result of a small displacement given to the system in its equilibrium configuration is an unbounded motion.

11.7 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Describe potential energy briefly.
2. Write a short notes on different kinds of equilibrium.

Long-Answer Questions

1. Describe motion of a one dimensional oscillator.
2. Show that the total energy of the oscillator is a constant of motion being proportional to the square of the amplitude and to the square of the frequency of the oscillator.

11.8 FURTHER READINGS

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UNIT 12 NORMAL MODES

NOTES

Structure

12.0 INTRODUCTION

A pattern of motion in which all parts of a system move sinusoidally with the similar frequency and with a fixed phase relation is known as normal mode of the oscillating system. The free motion expressed by the normal modes occurs at the fixe frequencies. These fixed frequencies of the normal modes of a system are called its natural frequencies or resonant frequencies. Normal modes are orthogonal to each other. They can move independently and excitation of one mode will never produce motion of a different mode. In this unit you will study motion of the system of two coupled oscillators and arbitrary constants expected from the general solutions of the two second order differential equations. You will learn about normal coordinates and normal modes. You will investigate the nature of any particular normal mode, symmetric mode and anti-symmetric mode. Equations of motion in normal coordinates are also derived in this unit.

12.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain motion of the system of two coupled oscillators.
- Determine Value of the arbitrary constants expected from the general solutions of the two second order differential equations.
- Understand normal coordinates and normal modes.
- Examine symmetric mode and anti-symmetric mode.
- Deduce the equations of motion in normal coordinates

12.2 TWO COUPLED OSCILLATORS - NORMAL COORDINATES AND NORMAL MODES

NOTES

As an example of a coupled system, let us consider two harmonic oscillators each of mass m coupled together by a spring of spring constant k' as shown in Fig. 12.1. Let the spring constant of one oscillator be k_1, and that of the other be k_2.

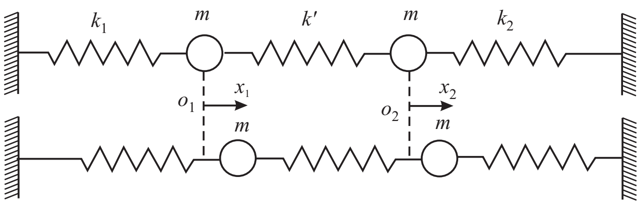

Fig. 12.1 *A Coupled System*

Let the motion of the two masses be constrained to take place along the line joining them, say along the X-axis.

The system thus has two degrees of freedom and the most convenient generalized coordinates may be taken as the displacements x_1 and x_2 of the two masses from their initial rest positions O_1 and O_2 at the instant of time t. We consider the displacements towards right as positive and that towards left as negative.

When the oscillators are not coupled by the spring, they vibrate independently of each other with frequencies

$$\omega_{10} = \sqrt{\frac{k_1}{m}}; \ \omega_{20} = \sqrt{\frac{k_2}{m}} \tag{1}$$

The kinetic energy of the coupled oscillators when they are displaced from their equilibrium positions is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 \tag{2}$$

The potential energy of the system is

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k'(x_1 - x_2)^2 \tag{3}$$

The Lagrangian of the system is thus

$$L = T - V = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 - \frac{1}{2}k'(x_1 - x_2)^2 \tag{4}$$

The Lagrange's equations of motion are

$$(i)\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \qquad (ii)\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

Using L as given by Equation (4), the two equations take the form

$$m\ddot{x}_1 + k_1 x_1 + k'(x_1 - x_2) = 0 \quad (5)$$

and
$$m\ddot{x}_2 + k_2 x_2 + k'(x_2 - x_1) = 0 \quad (6)$$

NOTES

The above two second-order differential equations may alternatively be written as

$$m\ddot{x}_1 + (k_1 + k')x_1 + k'x_2 = 0 \quad (7)$$

and
$$m\ddot{x}_2 + (k_2 + k')x_2 - k'x_1 = 0 \quad (8)$$

The above equations become independent of each other if the third term in each of these equations is not present. That is, if we hold the second mass at rest then $x_2 = 0$, and the frequency of oscillation of the first oscillator becomes

$$\omega_1' = \sqrt{\frac{k_1 + k'}{m}} \quad (9)$$

Similarly, if the first mass be held at rest, i.e., if $x_1 = 0$, then the frequency of vibration of the second oscillator becomes

$$\omega_2' = \sqrt{\frac{k_2 + k'}{m}} \quad (10)$$

We find $\omega_1' > \omega_{10}$ and $\omega_2' > \omega_{20}$. The reason for this is that each mass is attached to two springs and not one.

In order to obtain the different possible modes of vibration we must solve the simultaneous second-order differential Equation (7) and (8).

For simplicity, let us assume the two oscillators to be **exactly identical**. Then we take

$$k_1 = k_2 = k \quad (11)$$

so that the Equation (7) and (8) take the form

$$m\ddot{x}_1 + (k + k')x_1 - k'x_2 = 0 \quad (12)$$

$$m\ddot{x}_2 + (k + k')x_2 - k'x_1 = 0 \quad (13)$$

The trial solution of these equations can take any one of the following three forms:

$$x = A\cos(\omega t + \phi) \quad (14)$$

$$x = A_1 \cos \omega t + A_2 \sin \omega t \quad (15)$$

$$x = Ae^{\iota}(\omega t + \delta) \quad (16)$$

where δ is the initial phase factor. For generality, we assume Equation (16) to be in the form of the solution so that we get

$$x_1 = Ae^{i(\omega t + \delta_1)}\ ;\ x_2 = Be^{i(\omega t + \delta_2)} \quad (17)$$

NOTES

If we further assume the initial phase factors to be zero, i.e., if $\delta_1 = 0 = \delta_2$, then the solutions become

$$x_1 = Ae^{iwt} \tag{18}$$

$$x_2 = Be^{iwt} \tag{19}$$

Substituting the above solutions back into the Equation (12) and (13) and rearranging the terms we obtain

$$\left(k + k' - m\omega^2\right)A - k'B = 0 \tag{20}$$

and $$-k'A + \left(k + k' - m\omega^2\right)B = 0 \tag{21}$$

The above algebraic equations with the three unknowns A, B and ω upon solving give the ratio A/B as

$$\frac{A}{B} = \frac{k'}{k + k' - m\omega^2} = \frac{k + k' - m\omega^2}{k'} \tag{22}$$

From the above equality we can find ω. Alternatively, Equation (20) and (21) could be solved directly from the requirement that the determinant of the coefficients A and B is zero, i.e.

$$\begin{vmatrix} k + k' - m\omega^2 & -k' \\ -k' & k + k' - \omega^2 \end{vmatrix} = 0 \tag{23}$$

Equation (23) is called the secular equation, which gives

$$\left(k + k' - m\omega^2\right)^2 - k'^2 = 0$$

or $$\left(\frac{k}{m} - \omega^2\right)\left(\frac{k + 2k'}{m} - \omega^2\right) = 0 \tag{24}$$

The above yields the two roots

$$\omega^2 = \frac{k}{m} \text{ or } \omega = \pm\sqrt{\frac{k}{m}} = \pm\omega_1 \ (say) \tag{25}$$

and $$\omega^2 = \frac{k + 2k'}{m} \text{ or } \omega = \pm\sqrt{\left(\frac{k + 2k'}{m}\right)} = \pm\omega_2 \ (say) \tag{26}$$

In terms of the roots ω_1 and ω_2, the general solutions of the Equation (12) and (13) may be written as

$$x_1 = A_1e^{i\omega_1 t} + A_{-1}e^{-i\omega_1 t} + A_2e^{i\omega_2 t} + A_{-2}e^{-i\omega_2 t} \tag{27}$$

$$x_2 = B_1e^{i\omega_1 t} + B_{-1}e^{-i\omega_1 t} + B_2e^{i\omega_2 t} + B_{-2}e^{-i\omega_2 t} \tag{28}$$

In the above solutions, there are eight arbitrary constants. However, not all are independent. Substituting Equation (25) and (26) in Equation (20) and (21) or in Equation(22), we can obtain the ratios A/B for different values of ω. We also obtain

$$A = +B \text{ if } \omega = \omega_1 \tag{29}$$

and $$A = -B \text{ if } \omega = \omega_2 \tag{30}$$

Combining Equation (29) and (30) with Equation (27) and (28), we get

$$x_1 = A_1 e^{i\omega_1 t} + A_{-1} e^{-\omega_1 t} + A_2 e^{i\omega_2 t} + A_{-2} e^{-i\omega_2 t} \tag{31}$$

and

$$x_2 = A_1 e^{i\omega_1 t} + A_{-1} e^{-\omega_1 t} - A_2 e^{i\omega_2 t} - A_{-2} e^{-i\omega_2 t} \tag{32}$$

NOTES

Thus, we have only four arbitrary constants A_1, A_{-1}, A_2, and A_{-2} as expected from the general solutions of the two second-order differential equations. The values of three constants can be known from the initial conditions.

12.2.1 Normal Coordinates and Normal Modes

After substituting for four constants in Equation (31) and (32), we find that each coordinate (x_1 and x_2) depends on two different frequencies ω_1 and ω_2. This does not allow us to understand the type of motion of the system of coupled oscillators. It is, however, possible to find new coordinates X_1 and X_2 which are linear combinations of x_1 and x_2 such that each new coordinate oscillates with a single frequency. In the system of coupled oscillators that has been considered above, these new coordinates are simply the sum and difference of x_1 and x_2. We take

$$X_1 = x_1 + x_2 \tag{33}$$

Using Equation (31) and (32) in the above, we get

$$X_1 = 2\left(A_1 e^{i\omega_1 t} + A_{-1} e^{-\omega_1 t}\right) = Ce^{i\omega_1 t} + De^{-i\omega_1 t} \tag{34}$$

Similarly, taking

$$X_2 = x_1 - x_2 \tag{35}$$

and using Equation (31) and (32), we obtain

$$X_2 = x_1 - x_2 = 2\left(A_2 e^{i\omega_2 t} + A_{-2} e^{-\omega_2 t}\right) = Ee^{i\omega_2 t} + Fe^{-i\omega_2 t} \tag{36}$$

C, D, E and F appearing in Equation (34) and (36) are new constants.

We find from Equation (34) and (36) that while the new coordinate X_1 oscillates with frequency ω_1, the coordinate X_2 oscillates with frequency ω_2.

The coordinates X_1 and X_2 which oscillate with constant and definite frequencies are called normal coordinates and the corresponding oscillations are called normal modes of oscillation.

The new coordinates X_1 and X_2 correspond to new modes of oscillation. In each mode, oscillation takes place with a single frequency. These are called the normal modes and the corresponding coordinates are called the normal coordinates. One important feature of normal modes is that, for any given normal mode (here X_1 or X_2) all the coordinates (here x_1 and x_2) oscillate with the same frequency. In practice, all the normal modes are excited simultaneously. If however, some mode is not initially excited then the mode remains unexcited throughout the motion.

NOTES

The nature of any particular normal mode can be investigated if the other normal modes can be equated to zero. In the case under consideration, to study the appearance of X_1 mode

we must have $$X_2 = 0$$

or $$x_1 - x_2 = 0 \quad \text{or} \quad x_1 = x_2 \tag{37}$$

Thus, X_1 is a symmetric mode and as shown in the Fig. 12.2 both the masses have equal displacements, have the same frequency of oscillation $\omega_1 = \sqrt{\dfrac{k}{m}}$ and are in the same phase.

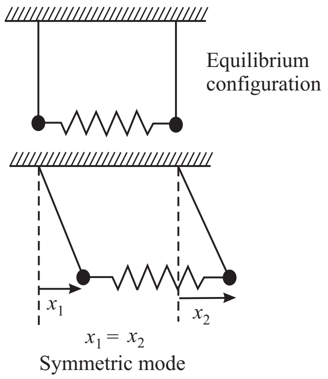

Fig. 12.2 *Symmetric Mode of Oscillation*

Similarly, to study the appearance of X_2 mode we need to let $X_1 = 0$ thereby requiring

$$x_1 + x_2 = 0 \quad \text{or} \quad x_1 = -x_2 \tag{38}$$

Clearly, X_2 is an anti-symmetric mode as shown in the Fig.12.3.

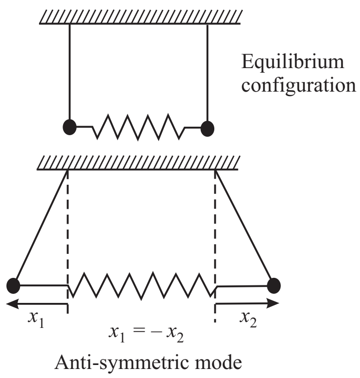

Fig. 12.3 *Anti-Symmetric Mode of Oscillation*

Both masses have equal but opposite displacements. Clearly, the masses are out of phase but they vibrate with the same frequency, $\omega_2 = \sqrt{\dfrac{k + 2k'}{m}}$.

We may summarize the above results as:

Symmetric mode $X_1, \omega_1 = \sqrt{\dfrac{k}{m}}, X_2 = 0, x_1 = x_2$

Anti-symmetric mode $X_2, \omega_2 = \sqrt{\dfrac{k+2k'}{m}}, X_1 = 0, x_1 = -x_2$

Conclusions

In the symmetric mode, the two coupled oscillators vibrate as if there were no coupling between them and their frequencies are the same as the original frequency.

In the anti-symmetric mode, the result of coupling is such that the oscillators oscillate out of phase with a frequency higher than their individual uncoupled frequencies.

In general, the mode that has the highest symmetry possesses the lowest frequency of vibration while the anti-symmetric mode has the highest frequency. In other words, as the symmetry is destroyed, the springs work harder causing the frequency to increase.

To excite the symmetric mode in the system under consideration, the two masses should be pulled from their equilibrium positions by equal amounts in the same direction and then released simultaneously.

To excite the anti-symmetric mode the two masses should be pulled from their equilibrium positions by equal amounts in *opposite* directions and then released simultaneously.

If the masses be given arbitrary displacements and then released, the motion of the system will consist of a combination of symmetric and anti-symmetric modes.

12.2.2 Equations of Motion in Normal Coordinates

We have the normal coordinates as

$$X_1 = x_1 + x_2$$

and $$X_2 = x_1 - x_2$$

The above relations give

$$x_1 = \frac{X_1 + X_2}{2} \tag{39}$$

and $$x_2 = \frac{X_1 - X_2}{2} \tag{40}$$

Substituting the above equations in Equation (2) and (3), we get

$$T = \frac{m}{2}\left(\frac{\dot{X}_1 + \dot{X}_2}{2}\right)^2 + \frac{m}{2}\left(\frac{\dot{X}_1 - \dot{X}_2}{2}\right)^2 = m\left(\frac{\dot{X}_1}{2}\right)^2 + m\left(\frac{\dot{X}_2}{2}\right)^2 \tag{41}$$

NOTES

$$V = \frac{k}{2}\left(\frac{X_1 + X_2}{2}\right)^2 + \frac{k}{2}\left(\frac{X_1 - X_2}{2}\right)^2 + \frac{k}{2}X_2^2$$

or
$$V = \frac{k}{2}\left(\frac{X_1^2}{2}\right) + \left(\frac{k + 2k'}{2}\right)\left(\frac{X_2^2}{2}\right) \qquad (42)$$

so that the Lagrangian in normal coordinates becomes

$$L = T - V = \frac{m}{4}\dot{X}_1^2 + \frac{m}{4}\dot{X}_2^2 - \frac{k}{4}X_1^2 - \left(\frac{k + 2k'}{4}\right)X_2^2 \qquad (43)$$

The Lagrange's equations in normal coordinates are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_1}\right) - \frac{\partial L}{\partial X_1} = 0$$

and
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}_2}\right) - \frac{\partial L}{\partial X_2} = 0$$

Using L given by Equation (43) in the above equations, we obtain

$$\ddot{X}_1 + \omega_1^2 X_1 = 0 \qquad (44)$$

where
$$\omega_1 = \sqrt{\frac{k}{m}} \qquad (45)$$

and
$$\ddot{X}_2 + \omega_2^2 X_2 = 0 \qquad (46)$$

where
$$\omega_2 = \sqrt{\frac{k + 2k'}{m}} \qquad (47)$$

Thus, an X_1 mode vibrates with frequency ω_1, while an X_2 mode vibrates with frequency ω_2 as seen in the previous section.

Check Your Progress

1. What is the kinetic energy of the coupled oscillators when they are displaced from their equilibrium positions?
2. What is the potential energy of the coupled oscillators when they are displaced from their equilibrium positions?
3. Write the Lagrangian of the coupled oscillators when they are displaced from their equilibrium positions?
4. Write the Lagrange's equations for the coupled oscillators when they are displaced from their equilibrium positions.
5. Define normal coordinates and normal modes.
6. How can we investigate the nature of any particular normal mode?
7. Differentiate between a symmetric an anti-symmetric mode.
8. How can we excite a symmetric mode?
9. What to do to excite an anti-symmetric mode?

12.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

NOTES

1. The kinetic energy of the coupled oscillators when they are displaced from their equilibrium positions is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

2. The potential energy of the system is

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k'(x_1 - x_2)^2$$

3. The Lagrangian of the system is thus

$$L = T - V = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2x_2^2 - \frac{1}{2}k'(x_1 - x_2)^2$$

4. The Lagrange's equations of motion are

$$(i)\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \qquad (ii)\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

5. The coordinates which oscillate with constant and definite frequencies are called normal coordinates and the corresponding oscillations are called normal modes of oscillation.
6. The nature of any particular normal mode can be investigated if the other normal modes can be equated to zero.
7. In the symmetric mode, the two coupled oscillators vibrate as if there were no coupling between them and their frequencies are the same as the original frequency.

   In the anti-symmetric mode, the result of coupling is such that the oscillators oscillate out of phase with a frequency higher than their individual uncoupled frequencies.
8. To excite the symmetric mode in the system under consideration, the two masses should be pulled from their equilibrium positions by equal amounts in the same direction and then released simultaneously.
9. To excite the anti-symmetric mode the two masses should be pulled from their equilibrium positions by equal amounts in *opposite* directions and then released simultaneously.

12.4 SUMMARY

- The kinetic energy of the coupled oscillators when they are displaced from their equilibrium positions is

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

NOTES

- The potential energy of the system is

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k'\ (x_1 - x_2)^2$$

- The Lagrange's equations of motion are

$$(i)\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \qquad (ii)\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0$$

- The coordinates which oscillate with constant and definite frequencies are called normal coordinates and the corresponding oscillations are called normal modes of oscillation.
- The mode that has the highest symmetry possesses the lowest frequency of vibration while the anti-symmetric mode has the highest frequency.

12.5 KEY WORDS

- **Normal coordinates and normal modes:** The coordinates which oscillate with constant and definite frequencies are known as normal coordinates and the corresponding oscillations are called normal modes of oscillation.
- **Symmetric mode:** In the symmetric mode, the two coupled oscillators vibrate as if there were no coupling between them and their frequencies are the same as the original frequency.
- **Anti-symmetric mode:** In the anti-symmetric mode, the result of coupling is such that the oscillators oscillate out of phase with a frequency higher than their individual uncoupled frequencies.
- **Oscillation:** It is the repetitive variation, usually in time of some measure about a central value (often a point of equilibrium) or between two or more distinct states. Common examples of oscillation involve a swinging pendulum and alternating current.

12.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Derive equations of motion in normal coordinates.
2. Give an example of a coupled system with suitable diagrams.
3. Write a short note on symmetric and anti-symmetric modes with suitable diagrams.

Long-Answer Questions

1. Describe motion of the system of two coupled oscillators.
2. Discuss normal coordinates and normal modes.

NOTES

12.7 FURTHER READINGS

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UNIT 13 GENERAL THEORY OF SMALL OSCILLATIONS

NOTES

Structure

13.0 INTRODUCTION

Oscillation is the repetitive variation, usually in time of some measure about a central value (generally a point of equilibrium) or between two or more distinct states. The term vibration is precisely used to express mechanical oscillation. Common examples of oscillation involve a swinging pendulum and alternating current. The characteristic equation is the equation derived by equating to zero the characteristic polynomial. It is known as a secular equation also which for linear systems is a method of finding the eigenvalues of the system. In this unit you will study general theory of small oscillations. You will examine secular equation and eigenvalue equation and understand kinetic energy of a system. Eigenvalue equation, Eigenvectors, and Eigen-frequencies are also discussed.

13.1 OBJECTIVES

After going through this unit, you will be able to:

- Explain general theory of small oscillators
- Distinguish a secular and an eigenvalue equation
- Understand kinetic energy of a system
- Calculate Eigenvalue equation, Eigenvectors, and Eigen-frequencies

NOTES

13.2 GENERAL THEORY OF SMALL OSCILLATIONS: SECULAR EQUATION AND EIGENVALUE EQUATION

Let us consider a system of n interacting particles with $3n$ degrees of freedom and described by a set of generalized coordinates $q_1, q_2, ..., q_{3n}$. Let us assume the force between the particles to be conservative so that the potential energy of the system can be expressed as

$$V = V(q_1, q_2, ..., q_{3n}) \tag{1}$$

Let us assume the system to undergo a small oscillation about an equilibrium configuration for which the generalized coordinates are $q_{10}, q_{20}, ..., q_{3n0}$. We may expand the potential energy about this equilibrium position in a multi-dimensional Taylor series as

$$V(q_1, ..., q_{3n}) = V(q_{10}, ..., q_{3N0}) + \frac{1}{1!}\sum_{l=1}^{3n}\left[\left(\frac{\partial V}{\partial q_l}\right)\Bigg|_{q_l=q_{l0}} (q_l - q_{l0})\right] \tag{2}$$

$$+\frac{1}{2!}\sum_{l=1}^{3n}\sum_{m=1}^{3n}\left[\left(\frac{\partial^2 V}{\partial q_l \partial q_m}\right)\Bigg|_{\substack{q_l=q_{l0}\\ q_m=q_{m0}}} (q_l - q_{l0})(q_m - q_{m0})\right] + ...$$

Since the zero of the potential energy is arbitrary, the first term in the above equation is a constant and may conveniently be taken equal to zero.

Further, since the system is in equilibrium, the generalized forces must vanish. We thus have

$$Q_1 = -\frac{\partial V}{\partial q_l} = 0; \quad l = 1, 2, ..., 3n \tag{3}$$

Clearly, the second term in Equation (2) also vanishes.

Thus, retaining the terms up to the second order, we obtain the potential energy as

$$V(q_1, ..., q_3 n) = \frac{1}{2!}\sum_{l=1}^{3n}\sum_{m=1}^{3n}\left[\left(\frac{\partial^2 V}{\partial q_l \partial q_m}\right)_{\substack{q_l=q_{l0}\\ q_m=q_{m0}}} (q_l - q_{l0})(q_m - q_{m0})\right] \tag{4}$$

Defining a new set of generalized coordinates η which represent displacements from the equilibrium configuration as

$$\eta = q - q_0 \tag{5}$$

we may write Equation (4) as

$$V = V(\eta_l) = \frac{1}{2!}\sum_{l=1}^{3n}\sum_{m=1}^{3n} V_{lm}\eta_l\eta_m \tag{6}$$

NOTES

where $$V_{lm} = \left(\frac{\partial^2 V}{\partial q_l \partial q_m}\right)\Bigg|_{\substack{q_l = q_{l0} \\ q_m = q_{m0}}} = V_{ml} = \text{constant} \qquad (7)$$

and $$\eta_1 = (q_1 - q_{10}) \text{ and } \eta_m = (q_m - q_{m0}) \qquad (8)$$

The constants V_{lm} form a symmetric matrix V. Since the motion is being considered about the stable equilibrium configuration, the potential energy must be the minimum, i.e., $V(\eta_l) > V(0)$. Hence, the homogeneous quadratic form of V given by Equation (6) must be positive. Thus, for a multi-dimensional system, the necessary and sufficient conditions that a homogeneous quadratic form of V be positive and definite are

$$\frac{\partial^2 V}{\partial q_l^2} > 0\,; \qquad l = 1, 2, \ldots.., 3n \qquad (9)$$

$$\begin{vmatrix} \dfrac{\partial^2 V}{\partial q_l^2} & \dfrac{\partial^2 V}{\partial q_l \partial q_m} \\ \dfrac{\partial^2 V}{\partial q_l \partial q_m} & \dfrac{\partial^2 V}{\partial q_m^2} \end{vmatrix} > 0; \qquad \begin{matrix} l = 1, 2, \ldots.., 3n \\ m = 1, 2, \ldots.., 3n \\ l \neq m \end{matrix} \qquad (10)$$

$$\begin{vmatrix} \dfrac{\partial^2 V}{\partial q_1^2} & \dfrac{\partial^2 V}{\partial q_1 \partial q_2} & - & \dfrac{\partial^2 V}{\partial q_1 \partial q_{3n}} \\ \dfrac{\partial^2 V}{\partial q_2 \partial q_1} & \dfrac{\partial^2 V}{\partial q_2^2} & - & \dfrac{\partial^2 V}{\partial q_2 \partial q_{3n}} \\ - & - & - & - \\ \dfrac{\partial^2 V}{\partial q_{3n} \partial q_1} & \dfrac{\partial^2 V}{\partial q_{3n} \partial q_2} & - & \dfrac{\partial^2 V}{\partial q_{3n}^2} \end{vmatrix} > 0 \qquad (11)$$

In terms of matrix notation the coefficients V_{lm} and V_{ml}, which are equal, must satisfy the conditions

$$V_{11} > 0$$

$$\begin{vmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{vmatrix} > 0$$

$$\begin{vmatrix} V_{11} & V_{12} & \ldots & V_{1m} \\ V_{21} & V_{22} & \ldots & V_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ V_{l1} & V_{l2} & \ldots & V_{lm} \end{vmatrix} > 0 \qquad (12)$$

where V_{lm} are given by Equation (7) and each individual V_{lm} need not be positive.

If the derivative $V_{lm} = \dfrac{\partial^2 V}{\partial q_l \partial q_m} = 0$ for all values of l and m, stable equilibrium is still possible provided the first non-zero derivative of the potential is of an even order.

NOTES

Kinetic Energy of the System

In terms of Cartesian coordinates, the kinetic energy is

$$T = \frac{1}{2}\sum_{j=1}^{n} m_j \dot{x}_j^2 \tag{13}$$

We have the transformation equation

$$x_j = x_j(q_1, ..., q_{3n}, t) \tag{14}$$

From Equation (14), we get

$$\dot{x}_j = \sum_{l=1}^{3n} \frac{\partial x_j}{\partial q_l}\dot{q}_l + \frac{\partial x_j}{\partial t} \tag{15}$$

Using Equation (15) in Equation (13), we obtain

$$T = \frac{1}{2}\sum_{j=1}^{n} m_j \left(\sum_{l=1}^{3n} \frac{\partial x_j}{\partial q_l}\dot{q}_l + \frac{\partial x_j}{\partial t}\right)\left(\sum_{m=1}^{3n} \frac{\partial x_j}{\partial q_m}\dot{q}_m + \frac{\partial x_j}{\partial t}\right) \tag{16}$$

Retaining only the terms which are quadratic in generalized velocities, the kinetic energy given by Equation (16) takes the form

$$T = \frac{1}{2}\sum_{l=1}^{3n}\sum_{m=1}^{3n}\left(\sum_{j=1}^{n} m_j \frac{\partial x_j}{\partial q_l}\frac{\partial x_j}{\partial q_m}\right)\dot{q}_l\dot{q}_m \tag{17}$$

For oscillations about the equilibrium configuration, we may write

$$\sum_{j=1}^{n} m_j \frac{\partial x_j}{\partial q_l}\frac{\partial x_j}{\partial q_m} = \sum_{j=1}^{n} m_j \left(\frac{\partial x_j}{\partial q_l}\right)_{q_{l0}}\left(\frac{\partial x_j}{\partial q_m}\right)_{q_{m0}} + \sum_{j=1}^{n} m_j \sum_{k=1}^{3n}\frac{\partial}{\partial q_k}\left(\frac{\partial x_j}{\partial q_l}\frac{\partial x_j}{\partial q_m}\right)_{\substack{q_{l0}\\ q_{m0}}} \eta_k + ... \tag{18}$$

where

$$\eta_k = q_k - q_{k0} \tag{19}$$

For small oscillations, we need to retain only those $\dot{q}$ terms in T which are of the same order as q in V.

Hence, from Equation (17), (18) and (19), we may write, remembering that $\dot{q}_l = \dot{\eta}_l$ and $\dot{q}_m = \dot{\eta}_m$.

$$T = \frac{1}{2}\sum_{l=1}^{3n}\sum_{m=1}^{3n} T_{lm}\dot{\eta}_l\dot{\eta}_m \tag{20}$$

In the above, we have

$$T_{lm} = \frac{1}{2}\sum_{j=1}^{n} m_j \left(\frac{\partial x_j}{\partial q_l}\right)_{q_{lo}}\left(\frac{\partial x_j}{\partial q_m}\right)_{q_{m0}} = T_{ml} \tag{21}$$

T_{lm} are the elements of a symmetric matrix T.

In view of Equation (6) and (20), the Lagrangian of the system becomes

$$L = T - V = \frac{1}{2}\sum_{l=1}^{3n}\sum_{m=1}^{3n}\left(T_{lm}\dot{\eta}_l\dot{\eta}_m - V_{lm}\eta_l\eta_m\right) \tag{22}$$

The Lagrange's equations given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\eta}}\right) - \frac{\partial L}{\partial \eta} = 0$$

NOTES

then take the form

$$\sum_{m=1}^{3n} \left(T_{lm}\ddot{\eta}_m + V_{lm}\eta_m\right) = 0; l = 1, 2, \ldots, 3n \tag{23}$$

or
$$T_{l1}\ddot{\eta}_1 + V_{l1}\eta_1 + T_{l2}\ddot{\eta}_2 + V_{l2}\eta_2 + \ldots + T_{l3n}\ddot{\eta}_{3n} + V_{l3n}\eta_{3n} = 0 \tag{24}$$

Equation (23) or (24) represent $3n$ linear, coupled, second-order differential equations which need to be solved to obtain the type of motion near the equilibrium configuration.

13.2.1 Eigenvalue Equation: Eigenvectors and Eigen-Frequencies

Since we are dealing with the oscillatory motion, the solution of Equation (23) may be taken in the form

$$n_j = Ca_j e^{iwt} \tag{25}$$

Here, Ca_j is the complex amplitude of oscillation corresponding to the coordinate η_j $(= q_j)$. The factor C is taken as a scale factor which is the same for all coordinates.

Using Equation (25) in Equation (23) we obtain

$$\sum_{j=1}^{3n}\left[T_{lj}\left(-\omega^2\right)Ca_j e^{i\omega t} + V_{lj}Ca_j e^{i\omega t}\right] = 0$$

or
$$\sum_{j=1}^{3n}\left[V_{lj}a_j - \omega^2 T_{lj}a_j\right]e^{i\omega t} = 0$$

The above gives

$$\sum_{j=1}^{3n}\left[V_{lj}a_j - \omega^2 T_{lj}a_j\right] = 0; \qquad (\because e^{iwt} \text{ is in general not zero}) \tag{26}$$

The above can be written in the form

$$V_a - \omega^2 Ta = 0 \tag{27}$$

where V, T and a are the matrices given by

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13n} \\ V_{21} & V_{22} & V_{23n} \\ V_{3n1} & V_{3n2} & V_{3n3n} \end{pmatrix} \tag{28}$$

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_{3n} \end{pmatrix} \text{ and } T = \begin{pmatrix} T_{11} & T_{12} & T_{13n} \\ T_{21} & T_{22} & T_{23} \\ T_{3n1} & T_{3n2} & T_{3n3n} \end{pmatrix}$$

NOTES

Equation (26) or Equation (27) gives 3n linear, homogeneous, algebraic equations which must be satisfied by a_j. For non-trivial solutions to exist, the determinant of the coefficients of a_j must vanish, i.e.,

$$\left|V_{ij} - \omega^2 T_{ij}\right| = 0$$

or
$$\begin{vmatrix} V_{11} - \omega^2 T_{11} & V_{12} - \omega^2 T_{12} & V_{13n} - \omega^2 T_{13n} \\ V_{21} - \omega^2 T_{21} & V_{22} - \omega^2 T_{22} & V_{23n} - \omega^2 T_{23n} \\ V_{3n1} - \omega^2 T_{3n1} & V_{3n2} - \omega^2 T_{3n2} & V_{3n3n} - \omega^2 T_{3n3n} \end{vmatrix} = 0 \qquad (29)$$

Since V_{ij} and T_{ij} are symmetric in the above, we have $V_{12} = V_{21}$, etc., and $T_{12} = T_{21}$, etc.

The Equation (29) is called the characteristic or the secular equation of the system. It is an equation of 3nth degree in ω^2. The equation can be solved to get 3n roots which can be labelled as $\omega_1^2, \omega_2^2, ..., \omega_{3n}^2$. The frequencies ω_k thus obtained are called eigen-frequencies of the system.

By substituting each root of the secular equation in Equation (26), the (3n – 1) values of a_j can be determined for each value of ω_k. Since there exist 3n values of ω_k, there occur 3n sets of values of a_j. Each of these sets of values of a_j corresponding to one value of ω_k defines the components of a 3n-dimensional vector $\vec{a}_k$. Such a vector $\vec{a}_k$ is called the **eigenvector** of the system. We find that $\vec{a}_k$ is the eigenvector belonging to eigen-frequency ω_k.

The j^{th} component of the k^{th} eigenvector is written as a_{jk}.

A general solution of Equation (26) is a superposition of oscillations with frequencies equal to the eigen-frequencies. Clearly, If the system is displaced slightly from the equilibrium configuration and then released, it undergoes oscillations of a small amplitude about the equilibrium configuration with frequencies ω_1, ω_2, ..., ω_{3n}.

The solutions of the secular Equation (29) are hence called the frequencies of free vibration or resonant frequencies of the system.

In view of the above results the most general solution given by Equation (25) can be expressed in the following form

$$q_j = \eta_j = \sum_{k=1}^{3n} f_k a_{jk} \cos(\omega_k t + \delta_k)$$

where the scale factor f_k and the phase factor δ_k can be determined using the initial conditions imposed on the system.

NOTES

Check Your Progress

1. Give the expansion of the potential energy about equilibrium position in a multi-dimensional Taylor series.
2. Show that the first term in the expansion of the potential energy about equilibrium position in a multi-dimensional Taylor series is a constant.
3. What is the kinetic energy in terms of Cartesian coordinates?
4. Give examples of 3n linear, coupled, and second order differential equations.
5. What is characteristic or secular equation?
6. When does a system undergo oscillations of a small amplitude about the equilibrium configuration?

13.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. Let us assume the system to undergo a small oscillation about an equilibrium configuration for which the generalized coordinates are q_{10}, q_{20}, ..., q_{3n0}. We may expand the potential energy about this equilibrium position in a multi-dimensional Taylor series as

$$V(q_1,.....,q_{3n}) = V(q_{10},.....,q_{3N0}) + \frac{1}{1!}\sum_{l=1}^{3n}\left[\left(\frac{\partial V}{\partial q_l}\right)\bigg|_{q_l=q_{l0}}(q_l - q_{l0})\right]$$

$$+\frac{1}{2!}\sum_{l=1}^{3n}\sum_{m=1}^{3n}\left[\left(\frac{\partial^2 V}{\partial q_l \partial q_m}\right)\bigg|_{\substack{q_l=q_{l0}\\ q_m=q_{m0}}}(q_l - q_{l0})(q_m - q_{m0})\right] + ...$$

2. We may expand the potential energy about this equilibrium position in a multi-dimensional Taylor series as

$$V(q_1,.....,q_{3n}) = V(q_{10},.....,q_{3N0}) + \frac{1}{1!}\sum_{l=1}^{3n}\left[\left(\frac{\partial V}{\partial q_l}\right)\bigg|_{q_l=q_{l0}}(q_l - q_{l0})\right]$$

$$+\frac{1}{2!}\sum_{l=1}^{3n}\sum_{m=1}^{3n}\left[\left(\frac{\partial^2 V}{\partial q_l \partial q_m}\right)\bigg|_{\substack{q_l=q_{l0}\\ q_m=q_{m0}}}(q_l - q_{l0})(q_m - q_{m0})\right] + ...$$

Since the zero of the potential energy is arbitrary, the first term in the above equation is a constant and may conveniently be taken equal to zero.

3. In terms of Cartesian coordinates, the kinetic energy is

$$T = \frac{1}{2}\sum_{j=1}^{n} m_j \dot{x}_j^2$$

4. The following equations represent 3n linear, coupled, second-order differential equations which need to be solved to obtain the type of motion near the equilibrium configuration:

$$\sum_{m=1}^{3n} (T_{lm}\ddot{\eta}_m + V_{lm}\eta_m) = 0;\ l = 1, 2, \ldots.., 3n$$

or $T_{l1}\ddot{\eta}_1 + V_{l1}\eta_1 + T_{l2}\ddot{\eta}_2 + V_{l2}\eta_2 + \ldots.. + T_{l3n}\ddot{\eta}_{3n} + V_{l3n}\eta_{3n} = 0$

5. It is an equation of 3nth degree in ω^2. The equation can be solved to get 3n roots which can be labelled as $\omega_1^2, \omega_2^2, \ldots.., \omega_{3n}^2$.

$$\begin{vmatrix} V_{11} - \omega^2 T_{11} & V_{12} - \omega^2 T_{12} & V_{13n} - \omega^2 T_{13n} \\ V_{21} - \omega^2 T_{21} & V_{22} - \omega^2 T_{22} & V_{23n} - \omega^2 T_{23n} \\ V_{3n1} - \omega^2 T_{3n1} & V_{3n2} - \omega^2 T_{3n2} & V_{3n3n} - \omega^2 T_{3n3n} \end{vmatrix} = 0$$

6. If the system is displaced slightly from the equilibrium configuration and then released, it undergoes oscillations of a small amplitude about the equilibrium configuration.

13.4 SUMMARY

- For a multi-dimensional system, the necessary and sufficient conditions that a homogeneous quadratic form of V be positive and definite are

$$\frac{\partial^2 V}{\partial q_l^2} > 0;\quad l \quad = \quad 1, 2, \ldots.., 3n$$

$$\begin{vmatrix} \dfrac{\partial^2 V}{\partial q_l^2} & \dfrac{\partial^2 V}{\partial q_l \partial q_m} \\ \dfrac{\partial^2 V}{\partial q_l \partial q_m} & \dfrac{\partial^2 V}{\partial q_m^2} \end{vmatrix} > 0;\quad \begin{matrix} l = 1, 2, \ldots.., 3n \\ m = 1, 2, \ldots.., 3n \\ l \neq m \end{matrix}$$

$$\begin{vmatrix} \dfrac{\partial^2 V}{\partial q_1^2} & \dfrac{\partial^2 V}{\partial q_1 \partial q_2} & - & \dfrac{\partial^2 V}{\partial q_1 \partial q_{3n}} \\ \dfrac{\partial^2 V}{\partial q_2 \partial q_1} & \dfrac{\partial^2 V}{\partial q_2^2} & - & \dfrac{\partial^2 V}{\partial q_2 \partial q_{3n}} \\ - & - & - & - \\ \dfrac{\partial^2 V}{\partial q_{3n} \partial q_1} & \dfrac{\partial^2 V}{\partial q_{3n} \partial q_2} & - & \dfrac{\partial^2 V}{\partial q_{3n}^2} \end{vmatrix} > 0$$

NOTES

- If the derivative $V_{\text{lm}} = \dfrac{\partial^2 V}{\partial q_l \partial q_m} = 0$ for all values of l and m, stable equilibrium is still possible provided the first non-zero derivative of the potential is of an even order.
- In terms of Cartesian coordinates, the kinetic energy is

$$T = \frac{1}{2}\sum_{j=1}^{n} m_j \dot{x}_j^2$$

- If the system is displaced slightly from the equilibrium configuration and then released, it undergoes oscillations of a small amplitude about the equilibrium configuration.

13.5 KEY WORDS

- **Kinetic energy:** The energy that an object possess due to its motion is known as kinetic energy.
- **Oscillatory motion:** Oscillatory motion means repeated motion in which an object repeats the same movement again and again. It is called periodic motion too. It is seen in pendulums, vibrating strings, and elastic materials such as a spring.
- **Eigenvalues:** They are a distinctive set of scalars related with a linear system of equations (i.e., a matrix equation). They are occasionally known as characteristic roots, characteristic values, proper values, or latent roots also.
- **Eigenvector:** It is a non-zero vector that varies by only a scalar factor when that linear transformation is applied to it.
- **Eigen-frequencies:** One of the natural resonant frequencies of a system is termed as Eigen-frequencies.
- **Secular equation:** The characteristic equation is known as a secular equation also which for linear systems is a method of finding the eigenvalues of the system.
- **Scale factor:** It is a number which scales or multiplies some quantity. It is sometimes referred to as sensitivity. The ratio of any two corresponding lengths in two identical geometric figures is also known as a scale factor.

13.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

NOTES

Short-Answer Questions

1. Briefly describe potential energy of a system undergoing small oscillations.
2. Write a short note on kinetic energy of a system undergoing small oscillations.
3. Describe Eigenvalue equation briefly.

Long-Answer Questions

1. Describe general theory of oscillations.
2. Deduce the secular or characteristic equation for a system undergoing small oscillations.
3. Discuss Eigenvalue equation explaining Eigenvectors and Eigen-frequencies.

13.7 FURTHER READINGS

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Goldstein, Herbert. 2011. *Classical Mechanics,* 3rd Edition. New Delhi: Pearson Education India.

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UNIT 14 LINEAR TRIATOMIC MOLECULE

NOTES

Structure

14.0 INTRODUCTION

Molecules composed of three atoms, of either the same or different chemical elements are known as triatomic molecules e.g., H_2O, and CO_2. Linear triatomic molecules owe their geometry to their sp or sp^3d hybridised central atoms. Most common examples of linear triatomic molecules are carbon dioxide and hydrogen cyanide. Xenon di-fluoride is one of the rare examples of a linear triatomic molecule possessing non-bonded pairs of electrons on the central atom. In this unit you will study about a linear triatomic molecular system discussing small oscillations in normal coordinates. You will understand vibrations of a linear symmetrical triatomic molecule. You will examine different cases of normal coordinates corresponding to the normal frequencies. Vibration of a diatomic molecule is also explained in this unit.

14.1 OBJECTIVES

After going through this unit, you will be able to:

- Understand a linear triatomic molecular system
- Discuss small oscillations in normal coordinates
- Describe vibrations of a linear symmetrical triatomic and diatomic molecule
- Examine different cases of normal coordinates corresponding to the normal frequencies

NOTES

14.2 SMALL OSCILLATIONS IN NORMAL COORDINATES - VIBRATIONS OF A LINEAR TRIATOMIC MOLECULE

14.2.1 Vibration of a Linear Symmetrical Triatomic Molecule

The generalities of the problem of the determination of normal modes are well illustrated by a simple classical model which gives a good representation of a linear triatomic molecular system.

Let us consider a linear symmetric triatomic molecule of the type YX_2 such as the CO_2 molecule. The Figure 14.1 shows the equilibrium configuration of the molecule. The central atom Y (labelled 2) of mass M is elastically coupled to two other atoms X and X (labelled 1 and 3) each of mass m. The elastic constant is k in each case and in equilibrium configuration the atoms are equally spaced on a straight line. We restrict our considerations to the motion along the line XYX only. We further assume the interaction between the end atoms 1 and 3 to be negligible.

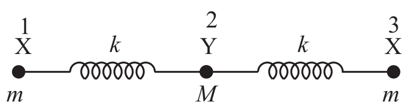


Fig. 14.1 Equilibrium Configuration of a Module

Let x_{1e}, x_{2e} and x_{3e} be respectively the distances of the atoms X, Y and X from an arbitrary fixed point on the straight line X Y X under the equilibrium condition. Let at any instant of time t, during the motion of the molecule x_1, x_2, x_3 be the distances of the atoms from the fixed point.

We then have the longitudinal displacements of the atoms X, Y and X from their equilibrium positions at the instant t

$$q_1 = x_1 - x_1e; \;\; q_2 = x_2 - x_{2e}; \;\; q_3 = x_3 - x_3e \qquad (1)$$

Let us choose q_1, q_2, q_3 defined above as the generalized coordinates describing the three atoms. We have the kinetic energy of the system.

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\,\dot{x}_3^2 + \frac{1}{2}M\dot{x}_2^2$$

In view of Equation (1) the above becomes

$$T = \frac{1}{2}m\dot{q}_1^2 + \frac{1}{2}m\dot{q}_3^2 + \frac{1}{2}M\dot{q}_2^2 \qquad (2)$$

The potential energy of the system is given by

$$V = \frac{1}{2}k(q_2 - q_1)^2 + \frac{1}{2}k(q_3 - q_2)^2 \qquad (3)$$

NOTES

The Lagrangian of the system is thus

$$L = T - V = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_3^2) + \frac{1}{2}M\dot{q}_2^2 - \frac{1}{2}k(q_2 - q_1)^2 - \frac{1}{2}k(q_3 - q_2)^2 \quad (4)$$

The Lagrange's equations of motion are

$$\begin{aligned} m\ddot{q}_1 &= k(q_2 - q_1) \\ M\ddot{q}_2 &= k(q_3 - q_2) - k(q_2 - q_1) = k(q_3 + q_1 - 2q_2) \\ M\ddot{q}_3 &= -k(q_3 - q_2) \end{aligned} \quad (5)$$

The assumption is now made that the motion is periodic. We can then write

$$q_k = q_{ke}e^{i\omega t}, \; k = 1, 2, 3 \quad (6)$$

where q_{ke} is the value of q_k under the equilibrium configuration.

Using Equation (6), Equation (5) gives

$$\begin{aligned} (m\omega^2 - k)q_{1e} + kq_{2e} &= 0 \\ (M\omega^2 - 2k)q_{2e} + k(q_{3e} + q_{1e}) &= 0 \\ (m\omega^2 - k)q_{3e} + kq_{2e} &= 0 \end{aligned} \quad (7)$$

The above equations are simultaneous equations in q_{1e}, q_{2e} and q_{3e}. For non-vanishing solutions, the determinant formed by the coefficients of q_{1e}, q_{2e} and q_{3e} must vanish.

We hence get

$$\begin{vmatrix} m\omega^2 k & k & 0 \\ k & M\omega^2 - 2k & k \\ 0 & k & m\omega^2 - k \end{vmatrix} = 0 \quad (8)$$

On expansion and algebraic simplification, the above gives

$$\omega^2(k - m\omega^2)\,[k(M + 2m) - \omega^2 Mm] = 0 \quad (9)$$

Solving the above we obtain

$$\omega^2 = 0, \; \frac{k}{m} \text{ and } \frac{k(M + 2m)}{Mm} \quad (10)$$

The **normal frequencies** are thus

$$\omega_1 = 0 \quad (11)$$

$$\omega_2 = \sqrt{\frac{k}{m}} \quad (12)$$

$$\omega_3 = \sqrt{\frac{k(M + 2m)}{Mm}} \quad (13)$$

The solution $\omega_1 = 0$ refers to translatory motion of the molecule on the whole.

The other two solutions refer to oscillatory motion of the molecule.

We may obtain Equation (8) alternatively as follows:

From Equation (2) we get

$$2T = m\dot{q}_1^2 + m\dot{q}_3^2 + M\dot{q}_2^2 \quad (14)$$

In matrix from the above can be written as

$$(2T) = (\dot{q}_1 \quad \dot{q}_2 \quad \dot{q}_3)\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} \quad (15)$$

The above gives the T matrix

$$T = (T_{ij})\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix} \quad (16)$$

From Equation (3) we get

$$2V = k(q_2 - q_1)^2 + k(q_3 - q_2)^2$$

which can be written in matrix form as

$$(2V) = (q_1 \quad q_2 \quad q_3)\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad (17)$$

The above gives V matrix as

$$V = (V_{ij}) = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \quad (18)$$

We have the secular determinant

$$| V - \omega^2 x | = 0$$

Using Equation (16) and (18) in the above we get

$$\begin{vmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0$$

The above is the same as Equation (8)

Normal Co-ordinates Corresponding to the Normal Frequencies

Let the normal coordinates corresponding to the normal frequencies ω_1, ω_2 and ω_3 be respectively Q_1, Q_2 and Q_3. Let $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ be the eigenvectors corresponding to the three modes of vibration respectively. To determine these eigenvectors we have the matrix equation

$$(V - \omega_k^2 T)(a_k) = 0 \quad (19)$$

NOTES

In the above (a_k) is the column matrix

$$(a_k) = \begin{pmatrix} a_1k \\ a_2k \\ a_3k \end{pmatrix} \tag{20}$$

Substituting Equation (6), (18) and (20) in Equation (19) we get

$$\begin{pmatrix} k - m\omega_k^2 & -k & 0 \\ -k & 2k - M\omega_k^2 & -k \\ 0 & -k & k - m\omega_k^2 \end{pmatrix} \begin{pmatrix} a_1k \\ a_2k \\ a_3k \end{pmatrix} = 0 \tag{21}$$

Case 1: $\omega_1 = 0$ $(k = 1)$

Equation (21) gives

$$\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \begin{pmatrix} a_11 \\ a_21 \\ a_31 \end{pmatrix} = 0$$

Carrying out the product we get

$$\begin{pmatrix} ka_{11} - ka_{21} \\ -ka_{11} + 2ka_{21} - ka_{31} \\ -ka_{21} + ka_{31} \end{pmatrix} = 0 \tag{22}$$

The above gives

$$ka_{11} - ka_{21} = 0 \text{ or } a_{11} = a_{21} \tag{23}$$

$$-ka_{21} + ka_{31} = 0 \text{ or } a_{21} = a_{31} \tag{24}$$

$$-ka_{11} + 2ka_{21} - ka_{31} = 0 \text{ or } -a_{11} + a_{21} - a_{31} = 0 \tag{25}$$

From Equation (23) and (24) we can write

$$a_{11} = a_{21} = a_{31} = \alpha(\text{say}) \tag{26}$$

Thus, for $\omega_1 = 0$, the eigenvector $\vec{a}_1$ is given by the matrix

$$\vec{a}_1 = (a_1) = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} \tag{27}$$

Case 2: $\omega_2 = \sqrt{\dfrac{k}{m}}$

We get from Equation (21)

$$\begin{pmatrix} k - m\dfrac{k}{m} & -k & 0 \\ -k & 2k - M\dfrac{k}{m} & -k \\ 0 & -k & k - m\dfrac{k}{m} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{23} \\ a_{32} \end{pmatrix} = 0$$

NOTES

or
$$\begin{pmatrix} 0 & -k & 0 \\ -k & 2k-\dfrac{M}{m}k & -k \\ 0 & -k & 0 \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = 0$$

δ
$$\begin{pmatrix} -ka_{22} \\ -ka_{12}+\left(2k-\dfrac{M}{m}k\right)a_{22}-ka_{32} \\ -ka_{22} \end{pmatrix} = 0$$

The above gives
$$a_{22} = 0 \tag{28}$$
$$a_{12} = -a_{32} = \beta \text{ (say)} \tag{29}$$

Thus for $\omega_2 = \sqrt{\dfrac{k}{m}},$

the eigenvector $\vec{a}_2$ is given by the column matrix
$$\vec{a}_2 = (a_2) = \begin{pmatrix} \beta \\ 0 \\ -\beta \end{pmatrix} \tag{30}$$

Case 3: $\omega_3 = \sqrt{\dfrac{k(M+2m)}{Mm}}$

Equation (21) gives
$$\begin{pmatrix} -k-\dfrac{k}{m}\left(1+\dfrac{2m}{M}\right)m & -k & 0 \\ -k & 2k-\dfrac{k}{m}\left(1+\dfrac{2m}{M}\right)m & -k \\ 0 & -k & k-\dfrac{k}{m}\left(1+\dfrac{2m}{M}\right)m \end{pmatrix}\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = 0$$

$$\begin{pmatrix} -k\dfrac{2m}{M} & -k & 0 \\ -k & k-k\dfrac{2m}{M} & -k \\ 0 & -k & \dfrac{-k2m}{M} \end{pmatrix}\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = 0$$

$$\begin{pmatrix} -k\dfrac{2m}{M}a_{13}-ka_{23} \\ -ka_{13}+\left(k-k\dfrac{2m}{M}\right)a_{23}-ka_{33} \\ -ka_{23}-k\dfrac{2m}{M}a_{33} \end{pmatrix} = 0 \tag{31}$$

NOTES

The above gives

$$\frac{2m}{M}a_{13} + a_{23} = 0 \text{ or } \frac{M}{2m}a_{23} = -a_{13}$$

$$-a_{13} + a_{23} - \frac{2m}{M}a_{23} - a_{33} = 0$$

$$a_{23} - \frac{2m}{M}\, a_{33} = 0 \text{ or } \frac{M}{2m}a_{23} = a_{33}$$

From the above we obtain

$$a_{13} = a_{33} = \gamma \text{ (say)} \tag{32}$$

$$a_{23} = \frac{-2m}{M}\gamma \tag{33}$$

Thus the eigenvector $\vec{a}_3$ corresponding to the normal frequency $\sqrt{\frac{k(M+2m)}{Mm}}$ is given by the column matrix

$$\vec{a}_3 = (a_3) = \begin{pmatrix} \frac{-2m}{M_\gamma}\gamma \end{pmatrix} \tag{34}$$

Using $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ as obtained above we get the eigenvector $\vec{a}$ $l_o - l_e$ the matrix

$$(a) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{35}$$

To calculate α, β and γ we use the orthogonality condition

$$(a) + \mathbf{T}(a) = 1$$

Using Equation (16) and (35) in the above we get

$$\begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & 0 & -\beta \\ \gamma & -\frac{2m}{M}\gamma & r \end{pmatrix}\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}\begin{pmatrix} \alpha & \beta & \gamma \\ \alpha & 0 & \frac{-2m}{M}\gamma \\ \alpha & -\beta & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{36}$$

On evaluating the left hand side of the above equation, Equation (36) becomes

$$\begin{pmatrix} \alpha^2(2m+M) & 0 & 0 \\ 0 & 2\beta^2 m & 0 \\ 0 & 0 & \gamma^2 2m\left(1+\frac{2m}{M}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Comparing the elements of the matrices on the two sides of the above equation we obtain

$$\alpha = \frac{1}{(2m+M)^{1/2}}, \quad \beta = \frac{1}{(2m)^{\frac{1}{2}}}, \quad \gamma = \frac{1}{\left\{2m\left(1+\frac{2m}{M}\right)\right\}^{1/2}} \tag{37}$$

The normal coordinates θ_1, θ_2 and θ_3 associated with the normal frequencies ω_1, ω_2, ω_3 respectively, are now obtained from

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{(2m+M)^{\frac{1}{2}}} & \frac{1}{(2m)^{\frac{1}{2}}} & \frac{1}{\left\{2m\left(1+\frac{2m}{M}\right)\right\}^{1/2}} \\ \frac{1}{(2m+M)^{\frac{1}{2}}} & 0 & \frac{-2m/M}{\left\{2m\left(1+\frac{2m}{M}\right)\right\}^{1/2}} \\ \frac{1}{(2m+M)^{\frac{1}{2}}} & -\frac{1}{(2m)^{\frac{1}{2}}} & \frac{1}{\left\{2m\left(1+\frac{2m}{M}\right)\right\}^{1/2}} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \tag{38}$$

14.2.2 Vibration of a Diatomic Molecule

A diatomic molecule may, in general, be asssumed to consist of two particles 1 and 2 having masses, say m_1 and m_2, respectively and interacting through a central potential $U(r_{12})$ which is a function of only the distance r_{12} between the particles.

With respect to an arbitrary fixed origin let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the Cartesian coordinates of the particles at some instant of time. The kinetic enegy of the system is then

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) \tag{39}$$

while the potential energy is

$$V = U(r_{12})$$

where
$$r_{12} = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2\right]^{1/2} \tag{40}$$

The Lagrangian of the molecule is thus

$$L = T - V = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2 + \dot{z}_2^2) - U(r_{12}) \tag{41}$$

If (r, θ, ϕ) be the relative spherical polar coordinates of the two particles, we have

$$\begin{aligned} x_1 - x_2 &= r\sin\theta\cos\phi \\ y_1 - y_2 &= r\sin\theta\sin\phi \\ z_1 - z_2 &= r\cos\theta \end{aligned} \tag{42}$$

NOTES

Further, if x, y, z be the coordinates of the centre of mass of the two particles we get

$$\begin{aligned} x &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \\ y &= \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \\ z &= \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2} \end{aligned} \qquad (43)$$

In terms of x, y, z and r, θ, ϕ the Lagrangian given by Equation (41) can be expressed as

$$L = \frac{1}{2} m_0 (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - U(r) \qquad (44)$$

where $m_0 = m_1 + m_2$ is the total mass of the molecule (45)

and $m = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass (46)

Under the equilibrium condition $U(r)$ is the minimum corresponding to equilibrium separation $r = r_0$ (say) = a constant. However, the remaining five coordinates x, y, z, θ and ϕ are arbitrary. Let these coordinates respectively have values x_0, y_0, z_0, θ_0 and ϕ_0 under the equilibrium configuration. We have six Lagrange's equations of motion and the eigenvalue equation is the 6×6 determinant

$$\begin{vmatrix} \omega^2 m_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega^2 m_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 m_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 m - \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega^2 r_0^2 m & 0 \\ 0 & 0 & 0 & 0 & 0 & m r_0^2 \omega^2 \sin^2 \theta_0 \end{vmatrix} = 0 \qquad (47)$$

In the above $\beta = \left(\frac{d^2 U}{dr^2}\right)_{r=r_0}$ (48)

On evaluating the determinant we obtain

$$(\omega^2)^5 m_0^3 \mu_0^2 r_0^4 \sin^2 \theta (\omega^2 m - \beta) = 0 \qquad (49)$$

The above equation is five-field degenerate and all the degenerate roots are zero, i.e.,

$$\omega_1^2 = \omega_2^2 = \omega_3^2 = \omega_5^2 = \omega_6^2 = 0 \qquad (50)$$

The only non-zero root is

$$\omega_4^2 = \frac{\beta}{m} \text{ or } \omega_4 = \sqrt{\frac{\beta}{m}} \qquad (51)$$

The normal coordinates are conveniently chosen as

$$Q_1 = \frac{x}{\sqrt{m_0}}, \qquad Q_2 = \frac{y}{\sqrt{m_0}}, \qquad Q_3 = \frac{z}{\sqrt{m_0}} \tag{52}$$

$$Q_5 = r_0\,\theta\sqrt{m}, \qquad Q_6 = r_0 \sin\theta_0\,\phi_0\sqrt{m} \tag{53}$$

$$Q_4 = r\sqrt{m} \tag{54}$$

NOTES

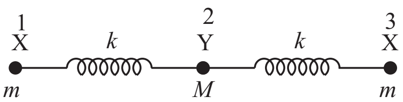
The three normal coordinates Q_1, Q_2 and Q_3 are cyclic and they correspond to translations.

The coordinates Q_5 and Q_6 are also cyclic corresponding to rotations. The coordinate Q_4 is non-cyclic and corresponds to vibration along the axis of the molecule.

Check Your Progress

1. Give diagrammatic representation of equilibrium configuration of a triatomic molecule.
2. What is translatory motion of the molecule on the whole?
3. What do you mean by oscillatory motion of the molecule?
4. What would be the matrix equation to determine eigenvectors for the vibrations of a linear triatomic molecule?
5. What is the eigenvector to the first mode of vibration?
6. Write the eigenvector to the second mode of vibration.
7. Write the eigenvector to the third mode of vibration.
8. What is a diatomic molecule?

14.3 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. 

2. The solution $\omega_1 = 0$ refers to translatory motion of the molecule on the whole.
3. The solutions $\omega_2 = \sqrt{\frac{k}{m}}$ and $\omega_3 = \sqrt{\frac{k(M+2m)}{Mm}}$ refer to oscillatory motion of the molecule
4. Let the normal coordinates corresponding to the normal frequencies ω_1, ω_2 and ω_3 be respectively Q_1, Q_2 and Q_3. Let $\vec{a}_1$, $\vec{a}_2$ and $\vec{a}_3$ be the

eigenvectors corresponding to the three modes of vibration respectively. To determine these eigenvectors we have the matrix equation $(V - \omega_k^2 T)(a_k) = 0$

5. For $\omega_1 = 0$, the eigenvector $\vec{a}_1$ is given by the matrix

$$\vec{a}_1 = (a_1) = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix}$$

6. $\omega_2 = \sqrt{\frac{k}{m}}$, the eigenvector $\vec{a}_2$ is given by the column matrix

$$\vec{a}_2 = (a_2) = \begin{pmatrix} \beta \\ 0 \\ -\beta \end{pmatrix}$$

7. the eigenvector $\vec{a}_3$ corresponding to the normal frequency $\sqrt{\frac{k(M+2m)}{Mm}}$ is given by the column matrix $\vec{a}_3 = (a_3) = \begin{pmatrix} \frac{-2m}{M_\gamma}\gamma \end{pmatrix}$

8. A diatomic molecule may, in general, be asssumed to consist of two particles 1 and 2 having masses, say m_1 and m_2, respectively and interacting through a central potential $U(r_{12})$ which is a function of only the distance r_{12} between the particles.

14.4 SUMMARY

- The solution $\omega_1 = 0$ refers to translatory motion of the molecule on the whole.
- To determine these eigenvectors we have the matrix equation $(V - \omega_k^2 T)(a_k) = 0$
- For $\omega_1 = 0$, the eigenvector $\vec{a}_1$ is given by the matrix

$$\vec{a}_1 = (a_1) = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix}$$

- A diatomic molecule may, in general, be asssumed to consist of two particles 1 and 2 having masses, say m_1 and m_2, respectively and interacting through a central potential $U(r_{12})$ which is a function of only the distance r_{12} between the particles.

14.5 KEY WORDS

- **Column matrix:** It is an ordered list of numbers written in a column. Each number of the column matrix is known as an element. The number of elements in a vector is known as its dimension.
- **Normal coordinates:** It is a set of coordinates for a coupled system such that the equations of motion each involve only one of these coordinates. Each normal coordinate identifies the instantaneous displacement of an independent mode of oscillation of the system.
- **Diatomic molecule:** Molecules composed of only two atoms, of the same or different chemical elements are called diatomic molecules, e.g., O_2, H_2, CO, and NO etc.
- **Triatomic molecule:** Molecules composed of three atoms, of either the same or different chemical elements are called triatomic molecules, e.g., O_3, H_2O, CO_2 and HCN.

14.6 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short-Answer Questions

1. Write a short note on equilibrium configuration of a triatomic molecule.
2. Give a brief account of cyclic coordinates corresponding to translations and rotations respectively. Also mention non-cyclic normal coordinates that corresponds to vibration along the axis of the molecule.
3. Briefly describe equilibrium configuration of a triatomic molecule.
4. Deduce the normal frequencies of a linear symmetric triatomic molecule.
5. Derive the eigenvector to the third mode of vibration.

Long-Answer Questions

1. Describe vibrations of a linear symmetrical triatomic molecule.
2. Explain different cases of normal coordinates corresponding to the normal frequencies.
3. Give a detailed account of vibration of a diatomic molecule.

NOTES

14.7 FURTHER READINGS

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